This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

CHAMBERS'S EDUCATIONAL COURSE,—EDITED BY W. AND R. CHAMBERS.

ELEMENTS
OF
PLANE GEOMETRY,
ACCORDING TO EUCLID,

AS IMPROVED BY

SIMSON AND PLAYFAIR;

WITH SEVERAL NEW IMPROVEMENTS AND ADDITIONS.

ndrew

BY A. BELL,

FORMERLY MATHEMATICAL MASTER IN DOLLAR INSTITUTION.

SECOND EDITION.

EDINBURGH:
PUBLISHED BY WILLIAM AND ROBERT CHAMBERS;
AND SOLD BY ALL BOOKSELLERS.
1837.

Educ 7018.37.205

HARVARD COLLEGE
July 22, 1909
LIBRARY

Graduate School of Education

EDINBURGH: PRINTED BY W. AND R. CHAMBERS,
19, WATERLOO PLACE.

PREFACE.

This treatise of Plane Geometry consists of the usual Six Elementary Books, a book on the Quadrature and Rectification of the Circle, a book on Geometrical Maxima and Minima, an exposition of the method of Geometrical Analysis, and an additional Second and Fifth Book. There is likewise added a Treatise on Plane Trigonometry, and an explanation of the Algebraical Principles employed in the fifth book.

The basis of the first six books is the Elements of Geometry of Euclid, as given in the very correct edition by Simson, with the improved fifth book by Playfair, and the other improvements of the latter geometer contained in his original edition of Euclid's Elements. None of the propositions of that edition are here suppressed, but proposition H of the sixth book is inserted at the end of the additional second book with a new demonstration, and instead of it a more important proposition is substituted at the end of the sixth book.

There are several improvements in this edition of the six books. Many additional and useful definitions have been added, which tend to improve the language of geometry in respect to conciseness and precision. Several propositions and corollaries have been inserted, as being valuable on account of their practical utility, or as rendering the treatise more complete. Two additional demonstrations are given of the fifth proposition of the first book, the common demonstration of which, from its tediousness, more than from any real difficulty, is rather an obstacle to the progress of young students. One of these demonstrations, against which I know of no objection, by the aid of an additional and admitted axiom, namely, the twelfth, is an easy and direct deduction from the fourth proposition. Numerous scholia have been added explanatory of the utility or connection of some of the propositions. The book on the quadrature of the circle, which is also given in Playfair's edition, is here considerably altered and improved. A new and rigorous demonstration is given of the tenth proposition of this book, which is of great utility in theory as well as in practice. The book on Geometrical Maxima and Minima is taken chiefly from Legendre's Geometry, with some alterations, particularly of the demonstration of the tenth proposition, which, in his treatise, is adapted to the particular view taken by him of the theory of proportion. An entirely new treatise of Plane Trigonometry has been composed for this edition.

For an account of the additional second and fifth books, reference is made to the introductory remarks to them. A considerable number of exercises are added after the several books, some of which are new, the most important of these being the 12th, 15th, and 23d of those at the end of the sixth book. The solution of exercises affords the best criterion of the industry and talents of the pupil; for in this exercise memory is of less service than usual, the inventive and reasoning faculties being chiefly exerted. In this matter, a few rules may be useful to the pupil, although he must be left chiefly to his own resources.

In order that nothing may be wanting to render the present treatise in every respect complete, an Introductory Discourse has been composed, affording an account of the chief objects of mathematical science, and which, with the History of Geometry which follows, will, it is hoped, tend to animate youth with a prospect of the advantages and pleasures to be derived from the prosecution of this branch of useful study.

Solid and Spherical Geometry, also Spherical Trigonometry, Conic Sections, &c. which compose the higher branch of Geometry, will form the matter of a subsequent volume.

EDINBURGH, *September* 1, 1836.

GEOMETRY.

INTRODUCTORY DISCOURSE.

THE study of mathematical science, of which Geometry is a branch, possesses several important advantages. Whether it be considered in reference to its practical utility, in its application to several important arts—or as a powerful, and the only adequate, instrument of investigation in the study of several classes of physical phenomena—or as an efficient instrument of intellectual culture—or merely in reference to the numerous and striking abstract truths which it makes known, it will, without hesitation, be admitted to be worthy of a prominent place in every course of useful and liberal education. The brief discussion of these points, with a sketch of the history of Geometry, will form the subject of this introduction.

OBJECTS OF MATHEMATICAL SCIENCE.

The science of mathematics investigates the various relations of measurable quantity. Space, time, force, velocity, and motion, are the principal objects about which it is conversant. Our knowledge of the objects of mathematics is obtained from experience, and its axiomatic principles are necessarily involved in our conceptions of these objects. Although the definitions of any of these objects are not necessarily confined to any single property, still every definition must express some characteristic property, and it cannot therefore be arbitrary.

Theoretical Geometry treats of the properties of magnitudes, and Practical Geometry of their construction. There are three kinds of magnitudes; they are of one, two, and three dimensions respectively, or lines, surfaces, and solids. Our conceptions of magnitude, and of space generally, are undoubtedly arrived at by first acquiring a knowledge of body by experience, and a subsequent process of abstraction. If we abstract from any body all the properties of the matter composing it, as its hardness, colour, weight, and so on, and retain merely its quality of extension in the three dimensions of length, breadth, and thickness, we have then a distinct conception of a geometrical solid, which possesses none of the properties of matter, except extension; and it therefore cannot possess a material existence. It is a different object from the space which it occupies; for in any limited portion of space, an indefinite number of such solids may exist, the one encompassing the other. Abstract now from any solid its thickness, and we then form the conception of a surface having only length and breadth. And if from a surface one of its dimensions be abstracted, as its breadth, we have then the idea of a line, which possesses only length. The intersection of two such lines is a point, which only marks position, and has neither length, breadth, nor thickness. It has been objected to this view of a mathematical point, that, as it has no magnitude, it can have no existence. It has certainly no material existence, but its existence is no less real on that account. Even a line or a surface occupies no portion of space. No number of points, however great, can fill any assignable portion of space, however small. And it has been remarked, that even a solid does not occupy exclusively any portion of space. They would exist independently of matter, but their existence is no less real, though immaterial.

A system of geometry proceeds from simple, axiomatic, and incontrovertible principles, to the demonstration of new truths; and from the combination of truths previously known, new truths are continually evolved; and thus a system of geometrical science is established by a continued process of logical deduction. Some of the elementary truths in geometry are so obvious as to be almost self-evident; but many of them are of a different character, and are striking, and even beautiful, at least when the mind is habituated to contemplate abstract truth. Some propositions are, in some of their cases, axiomatic, but in other cases they require to be demonstrated. Without this distinction, the demonstrations would appear to be unnecessary; and, in such cases, they are perhaps more useful in completely obviating objections, than in producing conviction.

UTILITY OF MATHEMATICS AS A MEANS OF EXTENDING OUR KNOWLEDGE OF THE MATERIAL WORLD.

Our knowledge has been greatly extended by means of this science. Independently of the innumerable important and striking properties of magnitudes and relations of abstract quantities that it has made known, and which can be sufficiently appreciated only by the mathematician, it has unfolded a very extensive range of natural phenomena. It has investigated the principles of theoretical mechanics; the laws of the equilibrium and motion of fluids, fixed and elastic; the principles of vision, of electricity, and of magnetism; the theory of the propagation of sound and of light; and a variety of other subjects. But even the most abstruse branches, that appear to be incapable of any useful application, ought not to be neglected; for they may be applied at some future period, like the ancient doctrine of the conic sections, which for two thousand years was an object of mere curious speculation, till it became, in the hands of Newton, a most efficient means of unfolding the planetary motions.

Without the aid of rules derived from this science, the navigator, relying only on his compass as a guide, could not with safety venture to any considerable distance on his element; intercourse with transmarine regions would be impossible; and, consequently, our knowledge of the globe which we inhabit would be very limited. We should probably still believe that its surface is an extended plane, and that it is supported on pillars; or, as was the opinion of some of the ancient philosophers, that its figure is cylindrical, like a drum. Without the aid of this science, our knowledge of celestial objects would be still more imperfect, and the consequences of our ignorance still more striking. We should still believe that these objects are equally distant from us, and, very probably, that they are distributed on the surface of an extensive crystalline sphere, performing a diurnal rotation about the earth, as the centre of the universe. We should also believe that some celestial phenomena, as eclipses and comets, are certain signs of a conflict of the elements of nature, or that they are the portentous indications of the wrath of heaven, while contemplating to inflict, on superstitious mortals, some dire calamity, as war, famine, or pestilence.

How different from these unsatisfactory and incoherent conjectures is that great achievement of this science—the clear and satisfactory exposition, on the most incontrovertible principles, of the complex though sublime and systematic mechanism of the heavens; by which the distances and magnitudes of the sun and planets have been measured, and also their weights, and even that of their satellites, ascertained; and by which the masses and distances of some of the stars or suns of other systems, though inconceivably remote, even in comparison with the great extent of our own system, will probably ere long be determined.

PRACTICAL UTILITY OF MATHEMATICS.

It will be unnecessary to make many observations on this subject, as the practical utility of mathematics is well known and indisputable. From its principles the rules of calculation and measurement are derived. It supplies the rules of the art of measuring distances, heights, surfaces, and solids, in artificers' work, guaging, land and marine surveying; it furnishes the principles of calculation in navigation, nautical and practical astronomy, of the arts of the optician and machinist, and also of the arts of carpentry and engineering, both civil and military. On its principles also depend the arts of planning, perspective, and of the construction of maps and charts. In short, whenever the construction of figures or computation is in requisition, the principles of mathematics are indispensable.

EFFICIENCY OF MATHEMATICS AS AN INSTRUMENT OF MENTAL IMPROVEMENT.

Considered as an instrument of intellectual improvement, it may be affirmed that mathematics cultivates chiefly the reasoning faculty. It also exercises the memory in a considerable degree; and it has a great tendency to form a habit of undivided and unremitting attention, which is indispensable for great success in any pursuit.

Every branch in the theory of the science consists almost entirely of an uninterrupted process of reasoning; and as this process is identical in every subject, whether of necessary or contingent truth, no other study can be more conducive to the improvement of this faculty. A step of reasoning, or a syllogism, consists of a major and minor proposition, and a conclusion; and, by a law of our mental constitution, whether it be called judgment or the faculty of relative suggestion, the conclusion follows as a necessary consequence from these premises, in reasoning in any subject as well as in mathematics; so that reasoning is exactly of the same nature in the investigation both of necessary and contingent truth—with this difference, that in the former the chain of reasoning is of almost indefinite extent, and in the latter it is generally brief. There is, however, a difference in the fundamental principles. The premises in the former are incontrovertible, at least in pure mathematics, and generally in the other branches of this science; whereas, in subjects of contingent matter, the premises are generally only probable, and the probability of the conclusion must therefore be commensurate with that of the premises.

Synthetic Geometry, or the ordinary didactic method, affords, in the gradual exposition of geometrical truth, excellent specimens of the most clear and satisfactory reasoning; and that branch of it called Geometrical Analysis, affords, in addition, examples of the resolution of truth into its simple elementary principles. But analytical geometry, and the other analytical branches of the science, supply the best examples of the resolution of complex questions—a process which must be effected before the conditions can be comprised in symbolical expressions; they also accustom the mind to comprehensive views, and afford excellent specimens of subtle reasoning; and exercise the mind in the interpretation of the expressions of the final result. In these branches, a subordinate acquirement, made at the expense of much perseverance, is necessary; namely, the power of managing skilfully the concise and comprehensive algorithm employed in its researches, of which, however, that part of the operations that may be considered to be in some measure mechanical, will sometimes interrupt the chain of reasoning, though in the theory the time thus spent by an expert analyst is comparatively small.

The application of the principles of the science to physical subjects, which constitutes the science of natural philosophy, in addition to the preceding kinds of intellectual exercise, affords examples of premises resting on probable evidence, and requires habits of close reflection and accurate observation, and also furnishes the finest specimens to be found in the whole range of human knowledge, of the methods of philosophical research, both inductive and deductive. In training the mind to such researches, it affords peculiar advantages; for although it is a subject of contingent matter, the rigorous nature of investigation operates as a salutary check against those fantastic speculations that result from the unrestrained excursions of the imaginative faculty, which, in original researches in other subjects, frequently produce extravagant theories, but which, from the unsettled state of the principles, may, with a little ingenuity, be made very plausible; whereas any such theory, in the former subject, would be certain to meet with speedy and complete refutation.

A knowledge of the methods of investigating necessary truth, is not inconsistent with a knowledge of the nature of moral evidence. An exclusive attention to any department of study, may to some extent disqualify the mind for appreciating truth in other departments. If the mere mathematician cannot appreciate minute degrees of moral evidence, neither can the mere student of probable truth appreciate the necessity of scientific rigour in mathematical science; and both might commit serious blunders in the department to which they are strangers; and the latter, if exclusively acquainted with those branches in which the premises are exceedingly doubtful, might, from the constant and bewildering uncertainty of his own conclusions, be liable to adopt a theory of universal scepticism. It is a truth readily assented to even by a mathematician, that of two contradictory propositions, that for which there is a preponderance of evidence, ought to be believed in preference to the other, although the amount of evidence fall far short of demonstration.

A step of reasoning in mathematics is clear and satisfactory when once perceived, which is also the case in other subjects; for in them the vagueness or unsatisfactoriness accompanying any discussion properly conducted, originates, not in the reasoning, but in the uncertainty, and sometimes the multiplicity, of the principles involved. A distinction, however, must be made between difficulty and uncertainty; for they are not necessarily connected, at least if difficulty be estimated by the degree of exercise required of the higher faculties. The converse of this, however, that is, the union of difficulty with certainty of principles, is constantly experienced by the mathematician; for, such is the complexity arising from the multiplicity of the principles involved in some subjects, that, notwithstanding the certainty of its principles, and the perfection of its language, and the almost magical powers of the higher calculus, they have baffled the most resolute efforts of the most able and vigorous minds; and had its language been less perfect, there are many subjects already thoroughly investigated, the difficulties of which would have been insurmountable.

It is an undoubted fact, that many men of reflecting minds have been devoted to mathematical study, which proves that there is an adaptation between it and minds of this complexion, or that it is fitted to afford their powers a sufficient exercise. Many celebrated mathematicians, too, have been very eminent for their attainments in general knowledge; in proof of which, it is merely necessary to mention the names of—Eratosthenes, of almost universal acquirements—the learned Beda—the eloquent Pascal—Ramus, of uncommon acuteness and eloquence—Descartes—Leibnitz—Condorcet—D'Alembert—Dr Clarke—Bishop Horsley—Playfair—and the all but universal Young; and the superior talents of many mathematicians, not so distinguished for varied attainments, is undeniable; as of Newton, Maclaurin, Lagrange, Laplace, and many others. In this science, too, there is great scope for the exercise of taste; for, since taste consists in the judicious selection of the fittest, and most agreeable, and most efficient means to accomplish an end, there must be an opportunity for its exercise in the discussion of scientific as well as of literary subjects; and the qualities of unity, clearness, force, and elegance, thus belong to scientific, as well as to literary, composition.

Mathematics, like any other science, cannot afford information respecting the principles of other subjects; but it possesses this peculiar advantage, that every branch of science tends rapidly towards a state of perfection in proportion as it admits of mathematical investigation. There is still a difference of opinion, not regarding the truth, but regarding the self-evidence, of some of the fundamental principles even of geometry, and also to what extent those of theoretical mechanics are dependent upon experience; and to investigate these, or the axiomatic principles of any science, or to appreciate moral evidence, a habit of reflection and of reasoning is indispensable, and the judgment must be the final arbiter. Since the science of theoretical mathematics consists almost entirely of a continued chain of reasoning, it affords, in a given period of study, many more examples of this process than any other subject. A mind, therefore, disciplined, though not exclusively, by this invigorating study, and also improved by the study of other branches, will certainly be the best qualified for investigating either necessary or contingent truth.

In a pamphlet entitled "Thoughts on the Study of Mathematics, as a part of Liberal Education, by the Rev. William Whewell, Cambridge," we find the following passages on the value of mathematics as a means of mental discipline. "In reasoning, as in other arts, we are not masters of what we have to do, till we do it both well and unconsciously. Now, this advantage a judicious cultivation of mathematics supplies. It familiarises the student with the usual forms of inference, till they find a ready passage through his mind; while any thing which is fallacious and logically wrong, at once shocks his habits of thought, and is rejected. He is accustomed to a chain of deduction, where each link hangs upon the preceding; and thus he learns continuity of attention and coherency of thought. His notice is steadily fixed upon those circumstances only in the subject on which the demonstrativeness depends; and thus that mixture of various grounds of conviction, which is so common in other men's minds, is rigorously excluded from his. He knows that all depends upon his first principles, and flows inevitably from them; that however far he may have travelled, he can at will go over any portion of his path, and satisfy himself that it is legitimate; and thus he acquires a just persuasion of the importance of principles, on the one hand, and, on the other, of the necessary and constant identity of the conclusions legitimately deduced from them."

HISTORY OF GEOMETRY.

The meaning of the term Geometry, in reference to its etymology, is the Measurement of the Earth. This was no doubt one of the original objects of the science of geometry; but its sphere soon became more widely extended, so that it is as applicable to the measurement of the heavens as to that of the earth.

Some of the elementary principles of geometry must have been known at a very remote period. Before a building of any considerable size could be erected, a plan of it must have been determined on, which could not be made without a knowledge of the simpler problems. This knowledge might have been attained, however, only by shrewd conjecture or tentative mechanical methods, instead of logical reasoning.

The opinions regarding the origin of geometry are various, but they concur in fixing it in Egypt. Some believe that it originated in the circumstance that the inundations of the Nile effaced the landmarks, and that it thus became necessary to assign annually to the proprietors their particular shares of land. The supposition of the obliteration of the landmarks, however, from this cause, is only an improbable conjecture. The historian Herodotus fixes its origin at the time when Sesostris intersected Egypt by numerous canals, and apportioned the country among the inhabitants; and Aristotle ascribes its origin to the Egyptian priests.

According to Plutarch, Thales of Miletus, who lived about six hundred years before the Christian era, measured the altitude of the pyramids of Egypt, and Amasis, the king, was astonished at his scientific skill; a fact which, if correct, proves the imperfect state of geometry at that time in Egypt. It is stated by Proclus that Thales could compute, by geometrical principles, the distance of vessels from the shore. Thales introduced the study of this science into Greece. He has the credit of making several discoveries of properties of the circle, and others respecting the comparison of triangles. The only discovery known to be his, however, is the proposition that all the angles in a semicircle are right angles. It is said, that, on the occasion of this last discovery, he offered in gratitude a sacrifice to the Muses. Thales founded a school in Greece, called the Ionian School.

The first elementary treatise of geometry is said to have been composed by Anaximander, a disciple of Thales, and his successor in his school. Anaximander was succeeded by Anaxagoras, who is the first recorded to have attempted the Quadrature of the Circle.

Pythagoras was another disciple of Thales. He travelled into Egypt and India in pursuit of knowledge. He settled in Tarentum, in Italy, where he founded the celebrated Pythagorean School (550 B.C.) He discovered the important proposition respecting the squares described on the sides of a right-angled triangle. On this occasion he is said to have sacrificed to the Muses a hundred oxen; a statement which is very improbable, both on account of the limited fortune of philosophers, and his philosophical creed respecting the transmigration of souls, which prohibited not only the destruction of animals, but even the use of animal food. He was also the first that observed that relation of lines called their incommensurability, and that invented the regular solids, afterwards called the Platonic bodies.

Hippocrates of Chios discovered the Quadrature of the Lune which has been called his lune, and he first proved that the solution of the problem of the Duplication of the Cube depends upon the finding of two mean proportionals; a question which arose from the request of the oracle at Delos to double the cubic altar of Apollo. He was the second who composed a treatise of geometry, which has been lost.

The origin of the Platonic School (400 B.C.) is considered to be one of the most important epochs in the history of the science. Its founder, Plato, is the reputed inventor of the method of Geometrical Analysis; and to this school we are indebted for the theory of the Conic Sections and of Geometrical Loci. These discoveries were probably the result of researches instituted for a very different purpose, namely, the duplication of the cube and the trisection of an angle—problems that were at this time of great celebrity; no less than eleven of the ancient geometers gave a solution of the former problem. The partiality of Plato for this science is evinced by his placing an inscription over the door of his school, forbidding those to enter who were ignorant of geometry.

Archytus of Tarentum, who belonged to the Pythagorean School, was the eighth successor of its founder, and the contemporary of Plato, whom he frequently visited. He solved the problem of two mean proportionals, and made several other discoveries by means of the geometrical analysis, which he learned from Plato. He was also well acquainted with the principles of mechanics. His flying pigeon is a proof of his reputation for mechanical invention, and he is said to be the inventor of the pulley and the screw. Leon of Neoclis, a philosopher, who was one of Plato's disciples, remodelled and extended the elements of geometry, to adapt them to the advanced state of the science, which had received considerable improvements from the time of Hippocrates. Eudoxus of Cnidus, a friend of Plato, and who is stated by Cicero to have been bred in the Egyptian School, is said by Archimedes to have discovered the contents of the pyramid and cone; and some consider him as the inventor of curve lines in general, and of the theory of proportion in Euclid's Elements of Geometry.

Menæchmus improved the Conic Sections, and gave two solutions of the problem of two mean proportionals. Dinostratus discovered a property of the Quadratrix, a curve which was no doubt invented for the purpose of solving the problem of the trisection of an angle, and the invention of which is ascribed to Hippias of Elis. Xenocrates, one of Plato's successors, composed works on arithmetic and geometry. Aristæus composed a treatise on Solid Loci, and another on Conic Sections. He had thus acquired great proficiency in what was called the sublime geometry; and he is said to have been the instructor of Euclid.

The Peripatetic School, founded by Aristotle (about 350

B. C.), was not so devoted as the Platonic to the study of mathematics. But Theophrastus, the successor of Aristotle, composed several mathematical works, and a complete history of the science, which have unfortunately been lost.

Geometry was cultivated with much attention in the Alexandrian school, which was established by Ptolemy Lagus (300 B. C.), who was a great patron of learned men. The celebrated geometrician Euclid, whose native place is unknown, but who had first studied geometry at Athens, went from Greece to Alexandria, where he settled during the time of the first Ptolemy. He composed treatises on various branches of mathematics, but his most eminent work is his Elements of Geometry. He was no doubt acquainted with the treatises of Hippocrates and Leon, but the fact of his treatise entirely supplanting theirs among the ancients, is a sufficient proof of their inferiority. Euclid has shown so much judgment in the composition of this treatise, that, notwithstanding the great additions made to geometry since his time, it has continued for upwards of two thousand years to sustain the highest reputation as an elementary treatise. He also composed a work called Data, another on Loci ad Superficiem, relating to curves of double curvature, and one on Porisms. The two latter have been lost; but, from the account given by Pappus, it appears that the last was a profound work relating to the analysis of the most abstruse and general problems.

Archimedes (born 287 B. C.) was the most celebrated of all the ancient mathematicians. He enjoyed the most extensive, and also the most popular reputation; for to his abstract researches he added several striking mechanical inventions; and he happened also to be placed in circumstances that made their value be appreciated. He began his career of geometrical discovery at that point at which ordinary minds are disposed to stop. He cultivated all the branches of mathematical science, but particularly the most difficult branches of geometry, relating to the areas of curves and the sections of curve surfaces. He employed, and with great address, in most of his discoveries, that subtle invention of the ancients, the Method of Exhaustions. He discovered that beautiful relation between a sphere and its circumscribing cylinder, that the capacity and surface of the former are respectively two-thirds of those of the latter; and also, that the surfaces of any two of their corresponding zones, formed by planes perpendicular to the axis of the cylinder, are equal. He also proved that the area of a circle is equal to *half* the rectangle contained by its circumference and radius, and he showed how to approximate as near as may be required to the quadrature of the circle, and found that of the parabola, which is memorable as the first example of the complete quadrature of a curve. He discovered several properties of conoids and spheroids, and of the spiral invented by Conon, but which, in honour of him who demonstrated its properties, is called the spiral of Archimedes. The writings of Archimedes prove to what an extent the sagacity of a mind like his could carry the ancient method of investigation; but they also prove the fact that there is a limit to its application. From the number and importance of his discoveries, though not to be compared to those of Newton, Archimedes has been called the Newton of antiquity.

Eratosthenes, a native of Cyrene, was called to the Alexandrian School by Ptolemy Euergetes, who made him his librarian on account of his extensive and general attainments in learning. He has given a construction for the duplication of the cube. His knowledge of geography and astronomy enabled him to discover, for the first time on record, a method for measuring the circumference of the earth.

Apollonius of Perga, in Pamphylia, was nearly a contemporary of Archimedes. He studied in the Alexandrian School in the time of Ptolemy Philopater. He greatly improved the conic sections, and composed several geometrical treatises, of which only one has reached us entire, namely, that on the Section of a Ratio; it is in Arabic, and was translated by Dr Halley. His other treatises were, on the Section of Space; on Determinate Section; on Tangencies; on Inclinations; on Plane Loci. He was an inventive and profound geometer, and his discoveries were so highly esteemed, that he was called the Great Geometer. The dispositions of Apollonius and Euclid were very different; the former was remarkable for vanity, jealousy of the merits of others, and detraction; the latter, for mildness, modesty, and benevolence.

Passing over the names of several geometers of little note, who lived about this period, we may merely mention Theodosius (50 B. C.), who composed an excellent work on Spherics, which is considered to be one of the most valuable remains of the ancient geometry; and Dionysiodorus, who found a solution of the problem of Archimedes, to cut a hemisphere in a given ratio by a plane parallel to its base, a problem that required considerable skill. Menelaus (second century) was the author of a treatise on trigonometry, and another on spherics.

Pappus and Theon of Alexandria (380 A.D.) were celebrated for their commentaries and annotations on the ancient geometry. To Pappus we are indebted for an excellent work, his Mathematical Collections, which consisted of eight books, of which the first and half of the second have perished. It is chiefly the elementary works of the ancient geometers that have been preserved; their more abstruse writings have either perished, or are only known by the accounts or abridgements of them that are given by Pappus. In this work he collected several detached original researches; and the preface to his seventh book is particularly valuable, as it has preserved several works on geometrical analysis, which would otherwise have been unknown. Hypatia, the accomplished and learned daughter of Theon, was so great a proficient in geometrical science, that she was considered qualified to be her father's successor in the Alexandrian School. But with all her merits, her fate was tragical, for she fell a victim to an infuriated and fanatical mob.

The philosopher Proclus, who was the head of the Platonic school, wrote a prolix commentary on the first book of Euclid, containing many curious observations on the metaphysics, and many interesting facts in the history of the science. Diocles invented the cissoid for the purpose of finding two mean proportionals, and Sporus gave a solution of this problem. Eutocius also gives the former the credit of solving the problem of Archimedes respecting the section of a hemisphere.

The age of important discovery was long past before the time of Pappus, and science in his time was beginning to decline. The labours of Proclus and a few of his followers were the last expiring efforts made in Greece in the ancient geometry. The Alexandrian School still existed, and science might have continued to flourish, had not the conquest of Egypt by the Arabians not only given it a fatal blow in Alexandria, but also extinguished its now languid existence in Greece. When declining in Greece, the sciences had met with some encouragement in Arabia; but notwithstanding this respect for science, the conquerors, instigated perhaps more by a recently imbibed religious fanaticism than by barbarian ignorance, committed to destruction that precious repository of science and learning, which contained the accumulated treasures of ages, the Alexandrian library, which was deliberately consumed for the mean purpose of heating the publio baths (640 A. D.)

As a partial amends for so odious an outrage against civilisation, the Arabians, about a century after, began to cherish the sciences of geometry and astronomy, and thus afforded an asylum which preserved them from entire extinction. About the time of their learned prince Almamon, several of the most celebrated works on geometry were translated into Arabic. Geber Ben Aphla laid the foundation of the modern trigonometry. An elegant treatise on Mensuration was written by Bagdadin; and it appears from his treatise on Optics, that Alhazin was a superior geometer. They studied also the more sublime geometry, and improved trigonometry by introducing sines instead of chords, and reduced it to a convenient form.

Of the few Persian geometers, Nassir-Edin and Maimon-Reschid were the most distinguished. The latter composed a commentary on Euclid, and the former, on Euclid and Apollonius. Their learned men called geometry the difficult science. They did not make any improvement in geometry; neither did the Turks nor Hebrews, although they translated some of the elementary works.

In one of the ancient treatises on astronomy of the Hindoos, called the Surya Sidhánta, there is a correct system of trigonometry; and hence it is concluded that the Indians had cultivated geometry before the composition of this treatise. The Chinese were acquainted with the celebrated property of a right-angled triangle respecting the squares of its sides, but they made no progress in the science.

The Romans knew little more of geometry than the practical branch of measuring land. Varro, however, composed a treatise on geometry, and Cicero had some acquaintance with, and esteem for, the science. Vitruvius and Boetius had some knowledge of geometry; but the latter is better known for his Consolations of Philosophy.

The learned Beda, who lived about the beginning of the eighth century, was well acquainted with all the branches of mathematics. Under him Alcuin studied, who was well versed in mathematical science, and was the preceptor of Charlemagne. He was a native of Britain, where, about this time, more attention was given to the study of science than in any country in Europe.

For nearly two centuries after this period, the study of the science was extinct. No mathematician of any eminence appeared till the beginning of the fifteenth century, when, with the general revival of learning, this science was destined to commence a new and more splendid career.

About this period, considerable improvements were made in trigonometry by Purbach, and John Müller, a native of Konigsberg, and hence called Regiomontanus. The former, instead of the sexagesimal division of the radius, proposed a division into 600,000 parts; but the latter, instead of this proposal of his master and predecessor, adopted the preferable division into 10,000,000; and calculated sines, tangents, and secants, to seven places of figures for every minute of the quadrant.

Campanus of Navarre, about the beginning of the thirteenth century, translated Euclid. Lucus Pacciolus or De Burgo revised it about the end of the fifteenth; and a new translation was given of it in 1572, by Commandinus. Maurolycus of Messina translated several geometrical works from Greek and Arabic manuscripts; and he was also distinguished for his original works, especially an elegant treatise on the Conic Sections, deduced from the sections of a cone. Tartaglia demonstrated the rule for finding the area of a triangle when its three sides are given, which was known, however, to the ancients; and he cultivated algebra. Wolfius, with an affectation of scientific rigour, attempted to demonstrate the axioms of geometry; and Ramus, or La Ramée, an original metaphysical writer, composed several excellent works on geometry.

Vieta of Fontenay perfected the modern system of trigonometry, and was the author of the elegant doctrine of Angular Sections. His profound knowledge of geometry enabled him to restore the Tangencies of Apollonius, in his Apollonius Gallus. About the same period, several geometers attempted to restore portions of the works of Apollonius. Marsinus Ghetaldus endeavoured to restore the treatise on Inclinations, under the title of Apollonius Redivivus, and wrote a supplement to the similar work of Vieta. To the former work of Ghetaldus, a supplement was written by Alexander Anderson, a Scotch geometer, who was well versed in the geometrical analysis. Willibrod Snellius published, at the age of seventeen, an attempted restoration of the work on Determinate Sections, under the title of Apollonius Batavus. The approximation to the value of the circumference of a circle, its diameter being unity, was successively extended about the same time by several geometers. It was carried by Vieta to eleven figures; by Adrianus Romanus to seventeen; and by Ludolph Van Ceulen to thirty-six; which was verified by an improved method by Snell. Joseph Scaliger also attempted the quadrature of the circle, and believed that he had found a solution of it, but was refuted by Romanus. Scaliger, however, gained some credit by his simplification of the cases of trigonometry.

About the beginning of the sixteenth century, the Spanish and Portuguese geometers, John of Royas and Nonnius or Nunez, lived. The former was the inventor of a projection of the sphere; the latter invented a method of dividing lines, similar in principle to Vernier's, and was the first to introduce the Arabic system of algebra into Europe. About the same time, Wright, a skilful geometrician, invented his chart, improperly called Mercator's Chart. Germany at this time produced the three mathematicians Werner, Rheticus, and Pitiscus. The former endeavoured to restore a work of Apollonius on the section of a Ratio, and wrote a German translation of Euclid; Rheticus extended considerably the trigonometrical tables, but Pitiscus, with incredible patience, carried them to a greater extent. The Italian geometer, Lucus Valerius, determined the centre of gravity of conoids and spheroids, and of some of their segments; and Albert Girard determined the surfaces of spherical triangles and polygons.

Kepler was the first to conceive the theory of magnitudes being composed of indefinitely small elements of the same kind; as, for instance, that a circle is composed of the surfaces of an indefinite number of indefinitely small triangles, having the centre for their common vertex, and their indefinitely small bases on the circumference. By means of this theory, though not very satisfactory in regard to scientific rigour, he extended geometry considerably beyond the stage in which it was left by Archimedes. Guldinus applied a property of the centre of gravity, previously known, to determine the contents of solids of revolution. Cavallerius improved on the theory of Kepler, by his theory of Indivisibles. He conceived lines to be composed of points, surfaces of lines, and solids of surfaces. This method was the means of extending geometry a little farther, and may be considered as another step in addition to the method of exhaustions and indefinitely small elements, in paving the way for the more general and rigorously scientific method of the Fluxionary or Higher Calculus.

Galileo, so eminent for his discoveries in physical science, did little to improve geometry. He invented the Cycloid, called the Helen of geometry, on account of the controversies that originated in the discussion of its properties. Torricelli and Viviani were disciples of Galileo. The former proved the finite magnitude of the solid generated by the revolution of a hyperbola about its asymptote, and the latter was distinguished for the elegance of his geometrical writings. Gregory St Vincent, and Andrew Tacquet, were two Flemish mathematicians; the former prosecuted the quadrature of the circle, and the latter, the quadrature of the surfaces, and the contents of the segments of cylinders cut by planes, and of several solids formed by the revolution of segments of circles and of conic sections.

Pascal, a Frenchman (born 1623—died 1662), who was equally distinguished for his great literary and scientific attainments, was the author of several elegant researches respecting the properties of the cycloid; and at the age of sixteen he composed a complete and elegant treatise of conic sections. Roberval and Descartes discovered the quadrature of the cycloid, and Pascal discovered that of its segments, and also the contents of the solids generated by the revolution of the figure about its base and axis; and Sir Christopher Wren discovered the quadrature of their surfaces.

Descartes, another eminent Frenchman (1596—1650), who had a profound knowledge both of mathematical and metaphysical science, and was one of the most efficient agents in accomplishing that intellectual revolution which was in progress during his time, was the inventor of a method which produced a thorough and memorable improvement in the science of geometry. Before his time, the application of algebra to geometry, in the ordinary sense of the term, was known, as examples of it are to be found as early as the time of Regiomontanus, and in the works of Tartaglia and Bombelli, and especially in those of Vieta. But Descartes invented that more general mode of applying it, by which the nature of a curve, comprehended in some of its characteristic properties, is expressed by an algebraical equation; which is the foundation of Analytical Geometry. This branch of geometry, instead of investigating separately the properties of particular curves, discusses, by general methods, whole classes of curves; and the science, instead of being confined within the narrow limits imposed by the complexity and inadequacy of the ancient methods, is now of an indefinite extent.

Descartes had given a general method of drawing tangents to curves, but it was in many cases of difficult application; and, to the discovery of a preferable method, Fermat, Roberval, and Barrow, directed their attention. But the methods invented by them, though capable in many cases of simple and elegant applications, were frequently impracticable, and inapplicable to transcendental curves. The method of Barrow, which was very similar to that of Fermat, contained the germ of the Differential and Integral

Calculus, as Roberval's did that of the Fluxionary. These methods of drawing tangents, which, like the method of exhaustions, and those of Kepler and Cavallerius, involved the principle of limits and indefinitely small elements, with the theory of the Arithmetic of Infinites of Wallis, contained all the geometrical and analytical views which embodied the fundamental principles of the higher calculus, and led to that memorable invention; which alone is all that was now necessary for the perfection of the science of geometry.

Hüygens, a native of Holland (1629—1705), was a profound mathematician. He determined the surfaces of conoids and spheroids, a problem of such difficulty, that it was not previously attempted. He also discovered the relation of Involutes and Evolutes, and a method of reducing the problem of rectification to that of quadrature. James Gregory was a geometer of the first rank, and distinguished for the excellence of his geometrical writings. He first published an account of the Logarithmic Curve in a work containing theorems for the quadrature and rectification of curves and the contents of solids of revolution. The geometrical writings of Barrow possessed great excellence; his geometrical lectures, like some of the works of Gregory, were written partly in the ancient and partly in the modern style.

Although the modern geometry had made great progress in the time of Newton, and was very much improved by the Fluxionary Calculus, which he had invented, yet its principles had not been so clearly developed, nor reduced to that systematic state which would have enabled him to establish, in so symmetrical a form, his profound discoveries, as by the ancient method, although the new calculus was undoubtedly the means employed in making these discoveries; and this circumstance, with his predilection for the ancient geometry, accounts for his preference of a method, which, notwithstanding the occasional elegance of its results, is one of great and embarrassing complexity. So much were the principles of the new calculus involved in unnecessary metaphysical difficulties, that Bishop Berkeley, with some plausibility, questioned their accuracy in a work entitled the Analyst, which induced Maclaurin to establish its principles irrefragably; and the means which he employed for this purpose were the principles of the ancient geometry. But, since his time, these principles have been established in a more simple and equally satisfactory manner, independently of geometrical considerations.

The ancient geometry has been thus superseded by the modern analytical methods, in the higher researches in mathematics. Maclaurin and Robison, however, two worthy disciples of Newton, have been considered as his only successors, who have had great success in employing the same instrument of investigation. Matthew Stewart was also distinguished for the uncommon elegance of his geometrical works; although the partial success attending his application of the science to some of the more difficult researches in physical astronomy, proves the inefficiency of the method, even when aided by the finest genius. Dr Simson is celebrated for his excellent edition of Euclid's Elements of Geometry, and he also restored some of the works of Apollonius. Bishop Horsley also published a superior edition of Euclid's Elements, and also restored some of the works of Apollonius; and the Rev. Mr Lawson also gave English editions of some of the works of the same author, and likewise a treatise on the Geometrical Analysis. Playfair improved considerably Simson's edition of Euclid's Elements of Geometry, by introducing algebraical notation into the demonstrations of the fifth book, which, by this means, are rendered less prolix, and more clear and precise. Euclid's definition of proportion, which is by no means an obvious property of proportional quantities, with this disadvantage, and the additional one of prolixity, possesses the important advantage of admitting direct and rigorously correct demonstrations in the case of incommensurables; and, with this definition, it is perhaps impossible to make any further improvement, of any importance, on Playfair's demonstrations, which appear to be as clear and concise as the nature of the subject will admit. On the subject of geometrical analysis, Sir John Leslie has given a very systematic treatise; but his geometry of curve lines, though containing many interesting geometrical researches, may, in regard to some portions of the work, be considered as another unsuccessful attempt, though seconded by superior talents, to apply the ancient geometry to subjects beyond its sphere.

There are many modern mathematicians of great eminence, whose names are not mentioned in this historical sketch of geometry, as they have distinguished themselves in other departments of mathematical science. Of these the most celebrated are Leibnitz, Euler, the Bernouillis, Clairaut, D'Alembert, Lagrange, Young, and Laplace. An account of their improvements and discoveries, and of those of Newton, Maclaurin, T. Simpson, and others, will be found in the history of those departments of science to which they belong.

ELEMENTS OF PLANE GEOMETRY.

FIRST BOOK.

Mathematics is that branch of science which treats of Measurable Quantity.

Geometry is a branch of mathematics, which treats of that species of quantity called Magnitude; it is either Theoretical or Practical.

Theoretical Geometry investigates the relations and properties of magnitudes.

Magnitudes are of one, two, or three dimensions; as lines, surfaces, and solids. They have no material existence, but they may be represented by diagrams.

That branch of Geometry which refers to magnitudes described upon a plane, is called Plane Geometry.

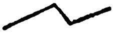
DEFINITIONS OF MAGNITUDES.

1. A *point* is that which has position, but not magnitude.
2. A *line* is length without breadth.

   COROLLARY.—The extremities of a line are points; and the intersections of one line with another are also points.

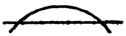
3. Lines which cannot coincide in two points, without coinciding altogether, are called *straight* or *right* lines.

   COR.—Hence two straight lines cannot inclose a space. Neither can two straight lines have a common segment; that is, they cannot coincide in part, without coinciding altogether.



4. A *crooked* or *broken* line is composed of two or more straight lines.

5. A line, of which no part is a straight line, is called a *curved line*, a *curve line*, or *curve*.

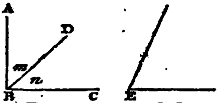


6. A *convex* or *concave* line is such that it cannot be cut by a straight line in more than two points; the *concavity* of the intercepted portion is turned towards the straight line, and the *convexity* from it.

7. A *superficies* is that which hath only length and breadth.

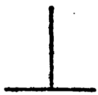
> COR.—The extremities of a superficies are lines; and the intersections of one superficies with another are also lines.

8. A *plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.



9. A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

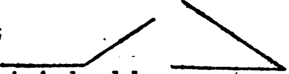
When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle—that is, at the point in which the straight lines that contain the angle meet one another—is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line: Thus the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, BD, is named the angle ABD, or DBA; and that which is contained by BD, CB, is called the angle DBC, or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E. It is sometimes convenient to name an angle by a small letter placed within the angle; thus, angle DBC may be called angle *n*; angle ABD, *m*; and angle ABC, *m n*.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle*; and the straight line which stands on the other is called a *perpendicular* to it.

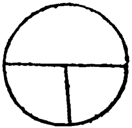
11. An *obtuse* angle is that which is greater than a right angle.

12. An *acute* angle is that which is less than a right angle.



13. A *figure* is that which is inclosed by one or more boundaries. The space contained within the boundary of a plane figure is called its *surface;* and its surface in reference to that of another figure, with which it is compared, is called its *area.*

14. A *circle* is a plane figure contained by one line, which is called the *circumference,* and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



15. This point is called the *centre* of the circle.

16. A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

17. A line drawn from the centre to the circumference of a circle is called a *radius.*

18. A *semicircle* is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. *Rectilineal* figures are those which are contained by straight lines.

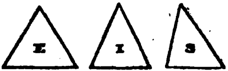
20. *Trilateral* figures, or triangles, by three straight lines.

21. *Quadrilateral,* by four straight lines.

22. *Multilateral* figures, or polygons, by more than four straight lines.

23. Of three-sided figures, an *equilateral* triangle is that which has three equal sides, as E.

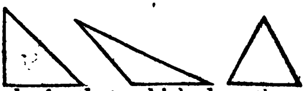
24. An *isosceles* triangle is that which has only two sides equal, as I.



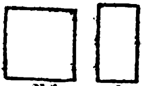
25. A *scalene* triangle is that which has three unequal sides, as S.

26. A *right-angled* triangle is that which has a right angle.

27. An *obtuse-angled* triangle is that which has an obtuse angle.

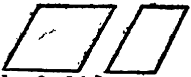


28. An *acute-angled* triangle is that which has three acute angles.



29. Of four-sided figures, a *square* is that which has all its sides equal, and all its angles right angles.

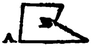
30. An *oblong* or *rectangle* is that which has all its angles right angles, but has not all its sides equal.



31. A *rhombus* is that which has all its sides equal, but its angles are not right angles.

32. A *parallelogram* is a quadrilateral, of which the opposite sides are parallel.

33. All other four-sided figures besides these, are called *Trapeziums.*



34. When an angle of a rectilineal figure is less than two right angles, it is called a *salient* angle; and when greater than two right angles, it is said to be *re-entrant*, as B.

35. Any side of a rectilineal figure may be called the *base.* In a right-angled triangle, the side opposite to the right angle is called the *hypotenuse;* either of the other two sides, the *base;* and the other, the *perpendicular.* In an isosceles triangle, the side which is neither of the equal sides is called the *base.* The point at which an angle is formed, is called the *angular* point. The angular point opposite to the base of a triangle is called the *vertex;* and the angle at the vertex, the *vertical* angle.

36. The *altitude* of a triangle, or a parallelogram, is a perpendicular drawn from the opposite angle or side upon the base.

37. A straight line joining two of the opposite angular points of a quadrilateral, is called a *diagonal.*

38. *Parallel* straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

POSTULATES.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, and with any radius.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are double of the same, are equal to one another.

7. Things which are halves of the same, are equal to one another.

8. Magnitudes which coincide with one another—that is, which exactly fill the same space—are equal to one another.

9. The whole is greater than its part.

10. All right angles are equal to one another.

11. Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.

12. It is possible for another figure to exist, equal in every respect to any given figure.

DEFINITIONS OF TERMS.

1. A *proposition* is a portion of science, and is a theorem, a problem, or a lemma.

2. A *theorem* is a truth which is established by a demonstration.

3. A *problem* either proposes something to be effected, as the construction of a figure; or it is a question that requires a solution.

4. A *lemma* is a subordinate truth previously established, to be employed in the demonstration of a theorem, or the solution of a problem.

5. A *hypothesis* is a fact assumed without proof, either in the enunciation of a proposition, or in the course of a demonstration.

6. A *corollary* is a consequence easily deduced from one or more propositions.

7. A *scholium* is a remark on one or more propositions, which explains their application, connection, limitation, extension, or some other important circumstance in their nature.

8. A *demonstration* is a process of reasoning, and is either direct or indirect.

9. A *direct* demonstration is a regular process of reasoning from the premises to the conclusion.

10. An *indirect* demonstration establishes a proposition, by proving that any hypothesis contrary to it, is contradictory or absurd; and it is, therefore, sometimes called a *reductio ad absurdum.*

11. The *enunciation* is the statement or expression of the proposition; the *particular* enunciation is its statement in reference to a particular figure or figures.

12. The *construction* is an operation by which lines are drawn or points determined according to certain conditions.

13. The *data* or *premises* of a proposition are the relations or conditions granted or given, from which new relations are to be deduced, or a construction to be effected.

EXPLANATION OF SIGNS.

The signs $+$ and $-$ are called *plus* and *minus.* The former indicates addition; thus $A + B$ is the sum of A and B, and $A + B + C$ is the sum of A, B, and C: the latter indicates subtraction; thus, $A - B$ is the excess of A above B; so $A + B - C$ is the excess of $A + B$ above C.

The signs $>$ and $<$ are called *greater* and *less.* Thus $A > B$ means that A is greater than B; and $A < B$, that A is less than B.

The sign $=$, called *equal,* is the sign of equality; thus $A = B$ implies that A is equal to B.

The small letters a, b, c, m, n, p, q, &c. are commonly used to denote numbers. A number placed before any quantity serves as a multiplier to it: thus, $3A$ means three times A; mA means m times A, or A taken as often as there are units in m.

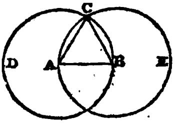
The square described on a line A is sometimes expressed by A^2 called A square; or, if AB be the line, by AB^2.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.

From the centre A, at the distance AB, describe (Postulate 3) the circle BCD, and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines (Post. 1) CA, CB, to the points A, B; ABC shall be an equilateral triangle.

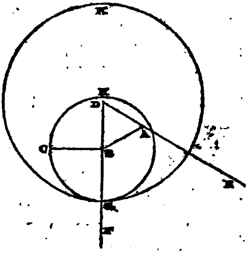


Because the point A is the centre of the circle BCD, AC is equal (Definition 14) to AB; and because the point B is the centre of the circle ACE, BC is equal to BA; but it has been proved that CA is equal to AB; therefore CA, CB, are each of them equal to AB; therefore (Axiom 1) CA is equal to CB; wherefore CA, AB, BC, are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB.

PROPOSITION II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

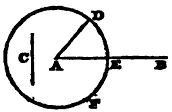
Let A be the given point, and BC the given straight line; it is required to draw from the point A a straight line equal to BC.—From the point A to B draw (Post. 1) the straight line AB; and upon it describe (I. 1) the equilateral triangle DAB, and produce (Post. 2) the straight lines DA, DB, to E and F; from the centre B, with the radius BC, describe (Post. 3) the circle CGH, and from the centre D, at the distance DG, describe the circle GKL. AL shall be equal to BC. Because the point B is the centre of the circle CGH, BC is equal (Def. 14) to BG; and because D is the centre of the circle GKL, DL is equal to DG, and



DA, DB, parts of them, are equal; therefore the remainder AL is equal to the remainder BG (Ax. 3): But it has been shown that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the same are equal to one another; therefore the straight line AL is equal to BC. Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.



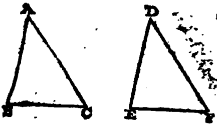
Let AB and C be the two given straight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.

From the point A draw (I. 2) the straight line AD equal to C; and from the centre A, and with the radius AD, describe (Post. 3) the circle DEF; and because A is the centre of the circle DEF, AE shall be equal to AD; but the straight line C is likewise equal to AD; whence AE and C are each of them equal to AD; wherefore the straight line AE is equal to C (Ax. 1), and from AB, the greater of two straight lines, a part AE has been cut off equal to C, the less.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, their bases, or third sides, shall be equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite. Or, if two sides and the contained angle of one triangle be respectively equal to those of another, the triangles are equal in every respect.

Let ABC, DEF, be two triangles which have the two sides AB, AC, equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF; and the angle BAC



equal to the angle EDF, the base BC shall be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal sides are opposite, shall be equal, each to each; namely, the angle ABC to the angle DEF, and the angle ACB to DFE.

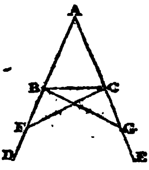
For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the straight line AB upon DE, the point B shall coincide with the point E, because AB is equal to DE; and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF; wherefore also the point C shall coincide with the point F, because AC is equal to DF: but the point B coincides with the point E; wherefore the base BC shall coincide with the base EF (Cor. Def. 3), and shall be equal to it. Therefore, also, the whole triangle ABC shall coincide with the whole triangle DEF, and be equal to it; and the remaining angles of the one shall coincide with the remaining angles of the other, and be equal to them; namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC, be produced to D and E, the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE, the greater, cut off AG equal (I. 3) to AF, the less, and join FC, GB.



Because AF is equal to AG, and AB to AC, the two sides FA, AC, are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the base FC is equal (I. 4) to the base GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite; namely, the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And because the whole AF is equal to the whole AG, and the part AB to the part AC, the remainder BF shall be equal (Ax. 3) to the remainder CG; and FC was proved to be equal to GB; therefore the two sides BF, FC, are equal to the two CG, GB, each to each; but the angle BFC is equal to the angle CGB; wherefore the triangles BFC, CGB, are equal, and their remaining angles are equal, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. Now, since it has been demonstrated that the whole angle ABG is equal to the whole ACF, and the part CBG to the part BCF, the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the base of the triangle ABC: And it has also been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base.

Cor.—Hence every equilateral triangle is also equiangular.

Scholium.—This proposition may be very simply demonstrated by bisecting the vertical angle A by a line cutting the base. Then (I. 4) there are two triangles equal in every respect, and therefore the angles at the base are equal. After Proposition 13 is proved, it could then be easily shown that the angles on the other side of the base are equal. It is evident that some line will bisect the vertical angle; and although the method of doing it is not known till Problem 9 be solved, yet this does not affect the truth of the proposition.

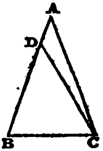
This proposition may also be easily demonstrated by supposing another isosceles triangle, having its two sides and vertical angle equal to those of ABC (Ax. 12). For if ABC, DEF, in Proposition 4, be these triangles, it is proved, as in Proposition 4, that angle B = E; but as in this case the two sides CA, AB, are also respectively equal to ED, DF, it follows that angle C = E (I. 4); and as B and C are each = E, therefore B = C.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to one another, the sides which subtend, or are opposite to, those angles, shall also be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the side AB is also equal to the side AC.

For, if AB be not equal to AC, one of them is greater than the other: Let AB be the greater, and from it cut (I. 3) off DB equal to AC, the less, and join DC; therefore, because in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two sides DB, BC, are equal to the two AC, CB, each to each; but the angle DBC is also equal to the angle ACB; therefore the base DC is equal to the base AB, and the triangle DBC is equal to the triangle ACB (I. 4), the less to the greater; which is absurd. Therefore AB is not unequal to AC; that is, it is equal to it.

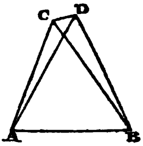


Cor.—Hence every equiangular triangle is also equilateral.

PROPOSITION VII. THEOREM.

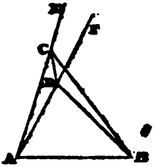
Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity, equal to one another.

If it be possible, let there be two triangles ACB, ADB, upon the same base AB, and upon the same side of it, which have their sides CA, DA, terminated in A equal to one another, and likewise their sides CB, DB, terminated in B, equal to one another.



Join CD; then, in the case in which the vertex of each of the triangles is without the other triangle, because AC is equal to AD, the angle ACD is equal (I. 5) to the angle ADC: But the angle ACD is greater than the angle BCD; therefore the angle ADC is greater also than BCD; still greater, then, is the angle BDC than the angle BCD. Again, because CB is equal to DB, the angle BDC is equal to the angle BCD; but it has been demonstrated to be greater than it; which is impossible.

But if one of the vertices, as D, be within the other triangle ACB; produce AC, AD, to E, F; then, because AC is equal to AD in the triangle ACD, therefore the angles ECD, FDC, upon the other side of the base CD, are equal to one another, but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewise greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, because CB is equal to DB, the angle BDC is equal to the angle BCD; but BDC has been proved to be greater than the same BCD; which is impossible.



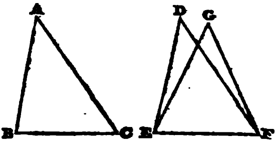
The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides of the other. Or, if the three sides of one triangle be respectively equal to those of another, the triangles are equal in every respect.

Let ABC, DEF, be two triangles having the two sides AB, AC, equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF; and also the base BC equal to the base EF. The angle BAC is equal to the angle EDF.

For, if the triangle ABC be applied to the triangle DEF, so that the point B be on E, and the straight line BC upon EF, the point C shall also coincide with the point F, because BC is equal to EF; and BC coinciding with EF, therefore BA and AC shall coincide with ED and DF; for, if BA and CA do not coincide with ED and FD, but have a different situation as EG and FG, then, upon the same



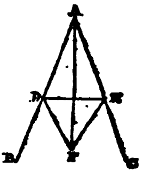
base EF, and upon the same side of it, there can be two triangles EDF, EGF, that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity. But this is impossible (I. 7); therefore, if the base BC coincides with the base EF, the sides BA, AC, cannot but coincide with the sides ED, DF; wherefore, likewise, the angle BAC coincides with the angle EDF, and is equal (Ax. 8) to it.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle—that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle; it is required to bisect it.

Take any point D in AB, and from AC cut (I. 3) off AE equal to AD; join DE, and upon it describe (I. 1) an equilateral triangle DEF; then join AF; the straight line AF bisects the angle BAC.



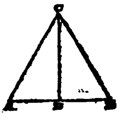
Because AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two sides DA, AF, are equal to the two sides EA, AF, each to each; but the base DF is also equal to the base EF; therefore the angle DAF is equal (I. 8) to the angle EAF; wherefore the given rectilineal angle BAC is bisected by the straight line AF.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line—that is, to divide it into two equal parts.

Let AB be the given straight line; it is required to divide it into two equal parts.

Describe (I. 1) upon it an equilateral triangle ABC, and bisect (I. 9) the angle ACB by the straight line CD. AB is cut into two equal parts in the point D.



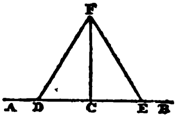
Because AC is equal to CB, and CD common to the two triangles ACD, BCD; the two sides AC, CD, are equal to the two BC, CD, each to each; but the angle ACD is also equal to the angle BCD; therefore the base AD is equal to the base DB (I. 4), and the straight line AB is divided into two equal parts in the point D.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a point given in it; it is required to draw a straight line from the point C at right angles to AB.

Take any point D in AC, and (I. 3) make CE equal to CD, and upon DE describe (I. 1) the equilateral triangle DFE, and join FC; the straight line FC, drawn from the given point C, is at right angles to the given straight line AB.

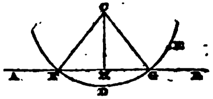


Because DC is equal to CE, and FC common to the two triangles DCF, ECF, the two sides DC, CF, are equal to the two EC, CF, each to each; but the base DF is also equal to the base EF; therefore the angle DCF is equal (I. 8) to the angle ECF; and they are adjacent angles; therefore (Def. 10) each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given straight line AB, FC has been drawn at right angles to AB.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a straight line perpendicular to AB from the point C.



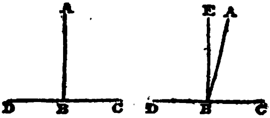
Take any point D upon the other side of AB, and from the centre C, with the radius CD, describe (Post. 3) the circle EGF meeting AB in F, G; and bisect (I. 10) FG in H, and join CF, CH, CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.

Because FH is equal to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC, are equal to the two GH, HC, each to each; now, the base CF is also equal (Def. 14) to the base CG; therefore the angle CHF is equal (I. 8) to the angle CHG; and they are adjacent angles; therefore each of them is a right angle, and CH is perpendicular to AB (Def. 10).

PROPOSITION XIII. THEOREM.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD; these are either two right angles, or are together equal to two right angles.



For, if the angle CBA be equal to ABD, each of them is a right angle (Def. 10); but, if not, from the point B draw BE at right angles (I. 11) to CD; therefore the angles CBE, EBD, are two right angles; and because CBE is equal to the two angles CBA, ABE, together, add the angle EBD to each of these equals; therefore the angles CBE, EBD, are equal (Ax. 2) to the three angles CBA, ABE, EBD. Again, because the angle DBA is equal to the two angles DBE, EBA, add to these equals the angle ABC; therefore the angles DBA, ABC, are equal to the three angles DBE, EBA, ABC; but the angles CBE, EBD, have been demonstrated to be equal to the same three angles; and things that are equal to the same are equal (Ax. 1) to one another; therefore the angles CBE, EBD, are equal to the angles DBA, ABC; but CBE, EBD, are two right angles; therefore DBA, ABC, are together equal to two right angles.

Or thus: let the angles DBE, EBA, ABC, be denoted by m, n, x, respectively (as in Def. 9), then DBA will be

mn, and EBC nx; also angle $mn = m + n$ and $nx = n + x$; therefore

$$mn + x = m + n + x;$$

but $m + nx = m + n + x$;

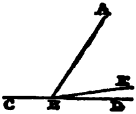
therefore $mn + x$ and $m + nx$, being equal to the same three angles, are equal to one another; that is,

$$mn + x = m + nx = \text{two right angles.}$$

PROPOSITION XIV. THEOREM.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD, equal together to two right angles. BD is in the same straight line with CB.

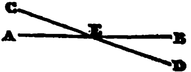


For, if BD be not in the same straight line with CB, let BE be in the same straight line with it; then, because the straight line AB makes angles with the straight line CBE, upon one side of it, therefore the angles ABC, ABE, are together equal (I. 13) to two right angles; but the angles ABC, ABD, are likewise together equal to two right angles; therefore the angles CBA, ABE, are equal to the angles CBA, ABD: take away the common angle ABC, and the remaining angle ABE is equal (Ax. 3) to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And, in like manner, it may be demonstrated, that no other can be in the same straight line with it but BD, which, therefore, is in the same straight line with CB.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical or opposite angles shall be equal.

Let the two straight lines AB, CD, cut one another in the point E; the angle AEC shall be equal to the angle DEB, and CEB to AED.



For the angles CEA, AED, which the straight line AE makes with the straight line CD, are together equal (I. 13) to two right angles; and the angles AED, DEB, which the straight line DE makes with the straight line AB, are also together equal to two right angles; therefore the two angles CEA, AED, are equal to the two AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal (Ax. 3) to the remaining angle DEB. In the same manner it can be demonstrated that the angles CEB, AED, are equal.

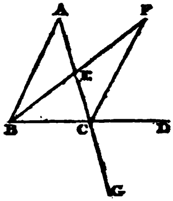
Cor. 1.—From this it is manifest, that, if two straight lines cut one another, the angles which they make at the point of their intersection, are together equal to four right angles.

Cor. 2.—And hence, all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D, the exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.



Bisect (I. 10) AC in E, join BE and produce it to F, and make EF equal to BE; join also FC, and produce AC to G.

Because AE is equal to EC, and BE to EF; AE, EB, are equal to CE, EF, each to each; and the angle AEB is equal to the angle CEF (I. 15), because they are opposite vertical angles; therefore (I. 4) the base AB is equal to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF; therefore the angle ECD, that is, ACD, is greater than BAE.

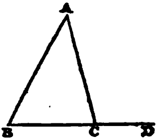
In the same manner, if the side BC be bisected, it may be demonstrated that the angle BCG, that is (I. 15), the angle ACD, is greater than the angle ABC.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle; any two of its angles together are less than two right angles.

Produce BC to D; and because ACD is the exterior angle of the triangle ABC, AGD is greater than the interior and opposite angle ABC (I. 16); to each of these add the angle ACB; therefore the angles ACD, ACB, are greater than the angles ABC, ACB; but ACD, ACB, are together equal to two right angles (I. 13); therefore the angles ABC, BCA are less than two right angles. In like manner it may be demonstrated, that BAC, ACB, as also CAB, ABC, are less than two right angles.

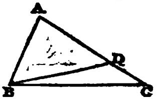


PROPOSITION XVIII. THEOREM.

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB; the angle ABC is also greater than the angle BCA.

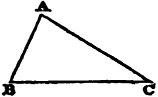
Because AC is greater than AB, make (I. 3) AD equal to AB, and join BD; and because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle DCB (I. 16); but ADB is equal to ABD (I. 5), because the side AB is equal to the side AD; therefore the angle ABD is likewise greater than the angle ACB; wherefore the angle ABC is still greater than ACB.



PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the side AC is likewise greater than the side AB.

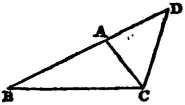


For, if it be not greater, AC must either be equal to AB, or less than it; it is not equal, because then the angle ABC would be equal to the angle ACB (I. 5); but it is not; therefore AC is not equal to AB; neither is it less; because then the angle ABC would be less than the angle ACB (I. 18); but it is not; therefore the side AC is not less than AB; and it has been shown that it is not equal to AB; therefore AC is greater than AB.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle; any two sides of it together are greater than the third side; namely, the sides BA, AC, greater than the side BC; and AB, BC, greater than AC; and BC, CA, greater than AB.



Produce BA to the point D, and make (I. 3) AD equal to AC; and join DC.

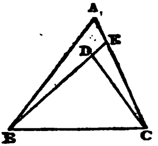
Because DA is equal to AC, the angle ADC is likewise equal to ACD (I. 5); but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC; and because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater side is opposite to the greater angle (I. 19); therefore the side DB is greater than the side BC; but DB is equal to BA and AC; therefore the sides BA, AC, are greater than BC. In the same manner it may be demonstrated, that the sides AB, BC, are greater than CA, and BC, CA, greater than AB.

PROPOSITION XXI. THEOREM.

If from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle, these two lines shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let the two straight lines BD, CD, be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it; BD and DC are less than the other two sides BA, AC, of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and because two sides of a triangle are greater than the third side, the two sides BA, AE, of the triangle ABE are greater than BE. To each of these add EC; therefore the sides BA, AC, are greater than BE, EC. Again, because the two sides CE, ED, of the triangle CED are greater than CD, add DB to each of these; therefore the sides CE, EB, are greater than CD, DB; but it has been shown that BA, AC, are greater than BE, EC; still greater, then, are BA, AC, than BD, DC.



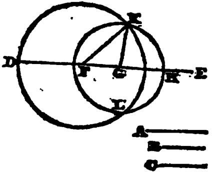
Again, because the exterior angle of a triangle is greater than the interior and opposite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the same reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle BDC is greater than the angle CEB; still greater, then, is the angle BDC than the angle BAC.

PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines; but any two whatever of these lines must be greater than the third (I. 20).

Let A, B, C, be the three given straight lines, of which any two whatever are greater than the third; namely A and B greater than C, A and C greater than B, and B and C than A. It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.

Take a straight line DE terminated at the point D, but unlimited towards E, and make (I. 3) DF equal to A, FG to B, and GH equal to C; and from the centre F, with the radius FD, describe the circle DKL (Post. 3); and from the centre G, with the radius GH, describe another circle HLK, and join KF, KG; the triangle KFG has its sides equal to the three straight lines A, B, C.

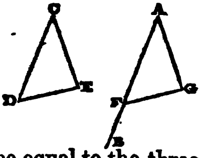


Because the point F is the centre of the circle DKL, FD is equal to FK (Def. 14); but FD is equal to the straight line A; therefore FK is equal to A. Again, because G is the centre of the circle LKH, GH is equal to GK; but GH is equal to C; therefore also GK is equal to C, and FG is equal to B; therefore the three straight lines KF, FG, GK, are equal to the three A, B, C. And therefore the triangle KFG has its three sides KF, FG, GK, equal to the three given straight lines, A, B, C.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.

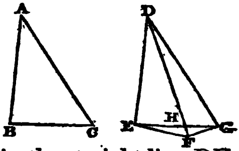


Take in CD, CE, any points D, E, and join DE; and make (I. 22) the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, CE, so that CD be equal to AF, CE to AG, and DE to FG; and because DC, CE, are equal to FA, AG, each to each, and the base DE to the base FG; the angle DCE is equal to the angle FAG (I. 8). Therefore, at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF, be two triangles which have the two sides AB, AC, equal to the two DE, DF, each to each; namely, AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the base BC is also greater than the base EF.



Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in the straight line DE, make (I. 23) the angle EDG equal to the angle BAC; and make DG equal to AC or DF (I. 3), and join EG, GF.

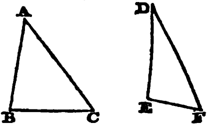
Because DE is not greater than DF or DG, therefore the angle DGE is not greater than DEG; but angle DHG is greater than DEG (I. 16); therefore DHG is greater than DGE, and hence the side DG or DF is greater than DH; therefore the line EG must lie between EF and ED, or the point F must be below EG.

Because AB is equal to DE, and AC to DG, the two sides BA, AC, are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the base BC is equal to the base EG (I. 4); and because DG is equal to DF, the angle DFG is equal to the angle DGF (I. 5); but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and still greater is the angle EFG than the angle EGF; and because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater side is opposite to the greater angle (I. 19), the side EG is therefore greater than the side EF; but EG is equal to BC; and therefore also BC is greater than EF.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

Let ABC, DEF, be two triangles which have the two sides AB, AC, equal to the two sides DE, DF, each to each; namely, AB equal to DE, and AC to DF; but the base CB is greater than the base EF; the angle BAC is likewise greater than the angle EDF.



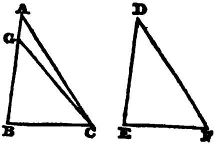
For, if it be not greater, it must either be equal to it, or less; but the angle BAC is not equal to the angle EDF, because, then, the base BC would be equal to EF (I. 4), but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it less; because then the base BC would be less than the base EF (I. 24); but it is not; therefore the angle BAC is not less than the angle EDF; and it was shown that it is not equal to it; therefore the angle BAC is greater than the angle EDF.

PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side; namely, either the sides adjacent to the equal angles, or the sides opposite to the equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.

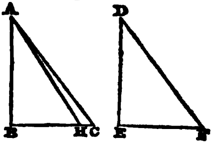
Or, if two angles and a side in one triangle be respectively equal to two angles and a corresponding side in another triangle, the two triangles shall be equal in every respect.

Let ABC, DEF, be two triangles which have the angles ABC, BCA, equal to the angles DEF, EFD; namely, ABC to DEF, and BCA to EFD; also one side equal to one side; and first let those sides be equal which are adjacent to the angles that are equal in the two triangles, namely, BC to EF; the other sides shall be equal, each to each, namely, AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, because BG is equal to DE, and BC to EF, the two sides GB, BC, are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal to the base DF (I. 4), and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypothesis, equal to the angle BCA; wherefore also the angle BCG is equal to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC, are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the base therefore AC is equal to the base DF, and the third angle BAC to the third angle EDF.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, namely, AB to DE; likewise in this case, the other sides shall be equal, AC to DF, and BC to EF; and also the third angle BAC to the third EDF.



For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; and because BH is equal to EF, and AB to DE, the two AB, BH, are equal to the two DE, EF, each to each; and they contain equal angles; therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore also the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle BCA; which is impossible (I. 16); wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC, are equal to the two DE, EF, each to each; and they contain equal angles;

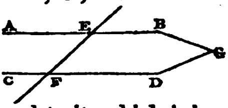
wherefore the base AC is equal to the base DF, and the third angle BAC to the third angle EDF.

PROPOSITION XXVII. THEOREM.

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another; AB is parallel to CD.

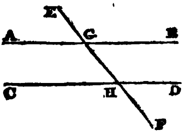
For, if it be not parallel, AB and CD, being produced, shall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D, in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater (I. 16) than the interior and opposite angle EFG; but it is also equal to it, which is impossible; therefore AB and CD being produced, do not meet towards B, D. In like manner it may be demonstrated, that they do not meet towards A, C; but those straight lines which meet neither way, though produced ever so far, are parallel to one another (Def. 38). AB therefore is parallel to CD.



PROPOSITION XXVIII. THEOREM.

If a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side; or make the interior angles on the same side BGH, GHD, together equal to two right angles; AB is parallel to CD.



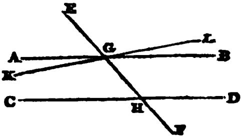
Because the angle EGB is equal to the angle GHD, and the angle EGB equal to the angle AGH (I. 15), the angle AGH is equal to the angle GHD; and they are the alternate angles; therefore AB is parallel to CD (I. 27). Again, because the angles BGH, GHD, are equal (by Hyp.) to two right angles; and that AGH, BGH, are also equal to two right angles (I. 13); the angles AGH, BGH, are equal to the angles BGH, GHD. Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD.

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles AGH, GHD, are equal to one another; and the exterior angle EGB is equal to the interior and opposite, upon the same side, GHD; and the two interior angles BGH, GHD, upon the same side, are together equal to two right angles.

For if AGH be not equal to GHD, let KG be drawn making the angle KGH equal to GHD, and produce KG to L; then KL will be parallel to CD (I. 27); but AB is also parallel to CD; therefore two straight lines are drawn through the same point G, parallel to CD, and yet not coinciding with one another, which is impossible (Ax. 11). The angles AGH, GHD, therefore are not unequal; that is, they are equal to one another. Now, the angle EGB is equal to AGH (I. 15); and AGH is proved to be equal to GHD; therefore EGB is likewise equal to GHD; add to each of these the angle BGH; therefore the angles EGB, BGH, are equal to the angles BGH, GHD; but EGB, BGH, are equal to two right angles (I. 13); therefore also BGH, GHD, are equal to two right angles.



Cor.—If two lines KL and CD make, with EF, the two angles KGH, GHC, together less than two right angles, KG and CH will meet on the side of EF on which the two angles are that are less than two right angles.

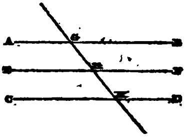
For, if not, KL and CD are either parallel, or they meet on the other side of EF; but they are not parallel; for the angles KGH, GHC, would then be equal to two right angles. Neither do they meet toward the points L and D; for the angles LGH, GHD, would then be two angles of a triangle, and less than two right angles; but this is impossible; for the four angles KGH, HGL, CHG, GHD, are together equal to four right angles, of which the two KGH, CHG, are by supposition less than two right angles; therefore the other two, HGL, GHD, are greater than two right angles. Therefore, since KL and CD are not parallel, and do not meet towards L and D, they will meet if produced towards K and C.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

Let AB, CD, be each of them parallel to EF, AB is also parallel to CD.

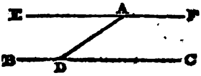
Let the straight line GHK cut AB, EF, CD; and because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (I. 29). Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD; and it was shown that the angle AGK is equal to the angle GHF; therefore also AGK is equal to GKD, and they are alternate angles; therefore AB is parallel to CD (I. 27).



PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw a straight line throught the point A, parallel to the straight line BC.



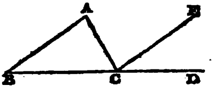
In BC take any point D, and join AD; and at the point A, in the straight line AD make (I. 23) the angle DAE equal to the angle ADC; and produce the straight line EA to F.

Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC, equal to one another, EF is parallel to BC (I. 27). Therefore the straight line EAF is drawn through the given point A parallel to the given straight line BC.

PROPOSITION XXXII. THEOREM.

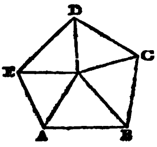
If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle; namely, ABC, BCA, CAB, are together equal to two right angles.



Through the point C draw CE parallel to the straight line AB (I. 31); and because AB is parallel to CE and AC meets them, the alternate angles BAC, ACE, are equal (I. 29). Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC; but the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC; to these angles add the angle ACB, and the angles ACD, ACB, are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB, are equal to two right angles (I. 13); therefore also the angles CBA, BAC, ACB, are equal to two right angles.

Cor. 1.—All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



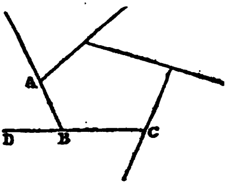
For any rectilineal figure ABCDE can

be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles; that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is (I. 15, Cor. 2), together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Let the sum of the interior angles be denoted by I, the number of sides by n, and a right angle by R, then

$$I + 4R = 2nR.$$

Cor. 2.—All the exterior angles of any rectilineal figure are together equal to four right angles.

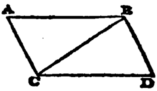


Because every interior angle ABC, with its adjacent exterior ABD, is equal to two right angles (I. 13); therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.

Let AB, CD, be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD; AC, BD, are also equal and parallel.



Join BC; and because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD, are equal (I. 29); and because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC, are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the base AC is equal to the base BD (I. 4), and the triangle ABC to the triangle BCD, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle ACB is equal to the angle CBD; and because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD, equal to one another, AC is parallel to BD (I. 27); and it was shown to be equal to it.

COR.—If two equal perpendiculars be drawn to a straight line on the same side of it, the straight line joining their extremities is parallel to the former, and equal to the intercepted part of it.

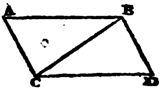
For the perpendiculars are equal and also parallel (I. 28); therefore the lines joining their extremities are equal and parallel.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of parallelograms are equal to one another, and the diagonal bisects them; that is, divides them in two equal parts.

Let ACDB be a parallelogram, of which BC is a diagonal; the opposite sides and angles of the figure are equal to one another, and the diagonal BC bisects it.

Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD, are equal to one another (I. 29); and because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD, are equal to one another; wherefore the two triangles ABC, CBD, have two angles ABC, BCA, in one, equal to two angles BCD, CBD, in the other, each to each, and one side BC common to the two triangles, which is adjacent to their equal angles; therefore their other sides shall be equal, each to each, and the third angle of the one to the third angle of the other (I. 26); namely, the side AB to the side CD, and AC to BD, and the angle BAC equal to the angle BDC: And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been shown to be equal to the angle BDC; there-



fore the opposite sides and angles of parallelograms are equal to one another: also, their diagonal bisects them; for AB being equal to CD, and BC common, the two AB, BC, are equal to the two DC, CB, each to each; and the angle ABC is equal to the angle BCD; therefore the triangle ABC is equal to the triangle BCD (I. 4), and the diagonal BC divides the parallelogram ACDB into two equal parts.

Cor. 1.—Parallel lines are equidistant.

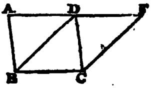
For, if from two points in one of them perpendiculars be drawn to the other, they are parallel (I. 28), and the two lines intercepted between them are parallel; therefore a parallelogram is formed, of which the perpendiculars are opposite sides, and are therefore equal.

Cor. 2.—Hence two triangles or two parallelograms between the same parallels have the same altitude; and if they have the same altitude, and be on the same side of bases that are in the same straight line, they are between the same parallels. (I. 33, Cor.)

PROPOSITION XXXV. THEOREM.

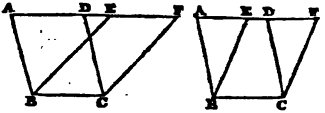
Parallelograms upon the same base and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF (see the 2d and 3d figures), be upon the same base BC, and between the same parallels AF, BC, the parallelogram ABCD shall be equal to the parallelogram EBCF.



If the sides AD, DF, of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain that each of the parallelograms is double of the triangle BDC (I. 34); and they are therefore equal to one another.

But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point, then, because ABCD is a parallelogram, AD is equal to BC; for the same reason EF is equal to BC; wherefore AD is equal to EF (Ax. 1); and DE is common; therefore (Ax. 2 or 3) the whole or the remainder AE is equal to the whole or the remainder DF; AB also is equal to DC; and the two EA, AB, are therefore equal to the two FD, DC, each to each; and the exterior angle FDC is equal (I. 29) to the interior EAB; therefore the base EB is equal to the base FC, and the triangle EAB equal to the triangle FDC (I. 4); take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB, the remainders therefore are equal (Ax. 3); that is, the parallelogram ABCD is equal to the parallelogram EBCF.

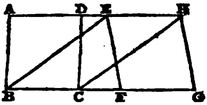


PROPOSITION XXXVI. THEOREM.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH, be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG, the parallelogram ABCD is equal to EFGH.

Join BE, CH; and because BC is equal to FG, and FG to EH (I. 34), BC is equal to EH; and they are parallels, and joined towards the same parts by the straight lines BE, CH; therefore (I. 33) EB, CH, are both equal and parallel, and EBCH is a parallelogram; and it is equal to ABCD (I. 35), because it is upon the same base BC, and between the same parallels BC, AH: For the like reason, the parallelogram EFGH is equal to the same EBCH; therefore also the parallelogram ABCD is equal to EFGH.

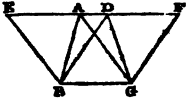


PROPOSITION XXXVII. THEOREM.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC, be upon the same base BC and between the same parallels AD, BC; the triangle ABC is equal to the triangle DBC.

Produce AD both ways to the points E, F, and through B draw BE parallel to CA (I. 31); and through C draw CF parallel to BD; therefore, each of the figures EBCA, DBCF, is a parallelogram; and EBCA



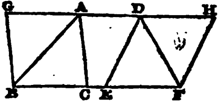
is equal to DBCF (I. 35), because they are upon the same base BC, and between the same parallels BC, EF; and the triangle ABC is the half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34); and the triangle DBC is the half of the parallelogram DBCF, because the diagonal DC bisects it. But the halves of equal things are equal (Ax. 7); therefore the triangle ABC is equal to the triangle DBC.

PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF, be upon equal bases BC, EF, and between the same parallels BF, AD; the triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and through B draw BG parallel (I. 31) to CA, and through F draw FH parallel to ED. Then each of the figures GBCA, DEFH, is a parallelogram; and they are equal to one another (I. 36), because they are upon equal bases BC, EF, and between the same parallels BF, GH; and the triangle ABC is the half of the parallelogram GBCA (I. 34), because the diagonal AB bisects it; and the triangle DEF is the half of the parallelogram DEFH, because the diagonal DF bisects it. But the halves of equal things are equal (Ax. 7); therefore the triangle ABC is equal to the triangle DEF.

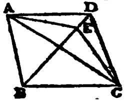


PROPOSITION XXXIX. THEOREM.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC, be upon the same base BC, and upon the same side of it; they are between the same parallels.

Join AD; AD is parallel to BC; for, if it is not, through the point A draw (I. 31) AE parallel to BC, and join EC; the triangle ABC is equal to the triangle EBC (I. 37), because it is upon the same base BC, and between the same pa-



rallels BC, AE. But the triangle ABC is equal to the triangle BDC; therefore, also, the triangle BDC is equal to the triangle EBC, the greater to the less, which is impossible; therefore AE is not parallel to BC. In the same manner, it can be demonstrated that no other line but AD is parallel to BC; AD is therefore parallel to it.

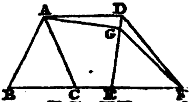
Cor.—This proposition is true of parallelograms.

For, if ABC, BDC, be the halves of the parallelograms, then, since AD is parallel to BC, the sides of the two parallelograms passing through A and D, and being both parallel to BC, must lie in the same line with AD (Ax. 11).

PROPOSITION XL. THEOREM.

Equal triangles upon the same side of equal bases, that are in the same straight line, are between the same parallels.

Let the equal triangles ABC, DEF, be upon the same side of equal bases BC, EF, in the same straight line BF; they are between the same parallels.



Join AD; AD is parallel to BC; for, if it is not, through A draw (I. 31) AG parallel to BF, and join GF. The triangle ABC is equal to the triangle GEF (I. 38), because they are upon equal bases BC, EF, and between the same parallels BF, AG; but the triangle ABC is equal to the triangle DEF; therefore also the triangle DEF is equal to the triangle GEF, the greater to the less, which is impossible; therefore AG is not parallel to BF. And in the same manner it can be demonstrated that there is no other parallel to it but AD; AD is therefore parallel to BF.

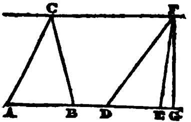
Cor.—This proposition is also true of parallelograms.

It may be proved in the same manner as the Cor. to last proposition.

PROPOSITION A. THEOREM.

Equal triangles, between the same parallels, are upon equal bases.

Let the areas of the triangles ABC, DEF, be equal, and let them lie between the same parallels AG, CF, then their bases AB, DE, are equal.



If AB be not equal to DE, let it be greater, and make DG = AB, and join G, F. Then, since AB = DG, the triangles ABC, DGF, are equal (I. 38); but the triangles ABC, DEF, are given equal; therefore the triangle DEF is equal to DGF, the less to the greater, which is impossible. Therefore the base AB is not greater than DE, and in a similar manner it may be proved not to be less; therefore AB = DE.

Cor.—This proposition is true of parallelograms.

For, if ABC, DEF, be the halves of the parallelograms, they are equal, and therefore AB = DE.

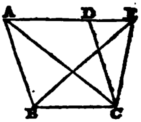
Scholium 1.—The preceding seven propositions, with the corollaries to the last three, are also true, when the term *altitude* is substituted for the expression *between the same parallels;* observing, that in the last three propositions it is not necessary that the two triangles or parallelograms be upon the same sides of their bases, or that their bases be in the same straight line; for when the two figures do not fulfil these conditions, other two equal to them in every respect may be constructed so as to fulfil them (I. 22), and the demonstrations will apply to the latter (I. 34, Cor. 2).

Schol. 2.—It appears from these propositions, also, that of these three conditions—the equality of the bases, of the altitudes, and of the areas of two parallelograms or triangles—if any two be given, the third will also be fulfilled: that is, if the bases and altitudes be equal, the areas are equal; if the bases and areas be equal, the altitudes are equal; and if the areas and altitudes be equal, the bases are equal.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC,



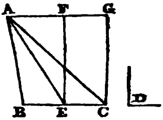
AE, the parallelogram ABCD is double of the triangle EBC.

Join AC; then the triangle ABC is equal to the triangle EBC (I. 37), because they are upon the same base BC, and between the same parallels BC, AE. But the parallelogram ABCD is double of the triangle ABC (I. 34), because the diameter AC divides it into two equal parts; wherefore ABCD is also double of the triangle EBC.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



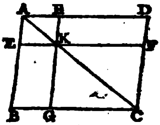
Bisect (I. 10) BC in E, join AE, and at the point E in the straight line EC make (I. 23) the angle CEF equal to D; and through A draw AG parallel to EC (I. 31), and through C draw CG parallel to EF; therefore FECG is a parallelogram. And because BE is equal to EC, the triangle ABE is likewise equal to the triangle AEC (I. 38), since they are upon equal bases BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle AEC. And the parallelogram FECG is likewise double of the triangle AEC (I. 41), because it is upon the same base, and between the same parallels; therefore the parallelogram FECG is equal to the triangle ABC, and it has one of its angles CEF equal to the given angle D; wherefore there has been described a parallelogram FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms which are about the diagonal of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is

AC; let EH, FG, be the parallelograms about AC; that is, through which AC passes, and BK, KD, the other parallelograms which make up the whole figure ABCD, which are therefore called the complements. The complement BK is equal to the complement KD.



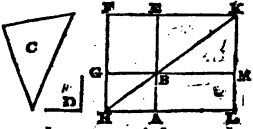
Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (I. 34). And because EKHA is a parallelogram, the diagonal of which is AK, the triangle AEK is equal to the triangle AHK. For the same reason, the triangle KGC is equal to the triangle KFC. Then, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC, is equal to the triangle AHK, together with the triangle KFC. But the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make (I. 42) the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, so that BE be in the same straight line with AB, and produce FG to H; and through A draw (I. 31) AH parallel to BG or EF, and join HB. Then because the straight line HF falls upon the parallels AH, EF, the angles AHF, HFE, are together equal to two right angles (I. 29); wherefore the angles BHF, HFE, are less than two right angles. But straight lines, which, with another straight line, make the interior angles upon the same side less than two right angles, do meet if produced far enough;

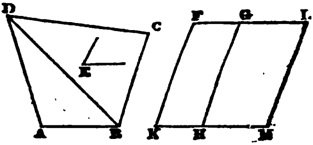


therefore HB, FE, shall meet, if produced; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB, to the points L, M. Then HLKF is a parallelogram, of which the diagonal is HK; and AG, ME, are the parallelograms about HK; and LB, BF, are the complements; therefore LB is equal to BF (I. 43). But BF is equal to the triangle C; wherefore LB is equal to the triangle C; and because the angle GBE is equal to the angle ABM (I. 15), and likewise to the angle D; the angle ABM is equal to the angle D; therefore the parallelogram LB is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

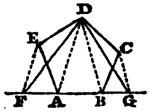


Join DB, and describe (I. 42) the parallelogram FH equal to the triangle ABD, and having the angle HKF equal to the angle E; and (I. 44) to the straight line GH apply the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E. And because the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of these the angle KHG; therefore the angles FKH, KHG, are equal to the angles KHG, GHM; but FKH, KHG, are equal to two right angles (I. 29); therefore also KHG, GHM, are equal to two right angles; therefore KH is in the same straight line (I. 14) with HM. And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF, are equal; add to each of these the angle HGL; therefore the angles MHG, HGL, are equal to the angles HGF, HGL; but the angles MHG, HGL, are equal to two right angles; wherefore also the angles HGF, HGL, are equal to two right angles, and FG is therefore in the same straight line with GL. And because KF is parallel to HG, and HG to ML, KF is parallel to ML (I. 30); but KM, FL, are parallels; wherefore KFLM is a parallelogram. And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM, the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E.

Cor.—From this it is manifest how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; namely, by applying to the given straight line a parallelogram equal to the first triangle ABD (I. 44), and having an angle equal to the given angle.

PROPOSITION B. PROBLEM.

To describe a triangle that shall be equal to a given rectilineal figure.



Let ABCDE be the given figure, to find a triangle equal to it.

Produce the side AB towards F and G; join D, B, and from C draw CG parallel to DB, and join D, G. Also, join A, D, and through E draw EF parallel to AD, and join D, F. FDG is the triangle required.

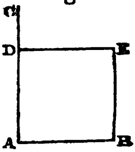
For, since DB and GC are parallel, the triangles DGB and DCB are equal (I. 37). For a similar reason, the triangles AED and AFD are equal; therefore the triangles AFD, BGD, together are equal to AED and BCD; and adding ADB to these equals, the whole triangle FDG is equal to the figure ABCDE.

Schol.—By means of this problem any crooked boundary may be rectified; that is, a straight line may be found that will cut off the same space on each side of it that the crooked boundary does. By means of the parallel ruler, this straight boundary may be easily found. The problem is of great utility in land-surveying.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Let AB be the given straight line; it is required to describe a square upon AB.



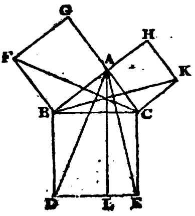
From the point A draw (I. 11) AC at right angles to AB, and make (I. 3) AD equal to AB, and (I. 31) through the point D draw DE parallel to AB, and through B draw BE parallel to AD; therefore ADEB is a parallelogram; whence AB is equal to DE (I. 34), and AD to BE. But BA is equal to AD; therefore the four straight lines BA, AD, DE, EB, are equal to one another, and the parallelogram ADEB is equilateral. It is likewise rectangular; for the straight line AD meeting the parallels AB, DE, makes the angles BAD, ADE, equal to two right angles (I. 29). But BAD is a right angle; therefore also ADE is a right angle. Now, the opposite angles of parallelograms are equal; therefore each of the opposite angles ABE, BED, is a right angle; wherefore the figure ADEB is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line AB.

Cor.—Hence every parallelogram that has one right angle has all its angles right angles.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle having the right angle BAC; the square described upon the side BC is equal to the squares described upon BA, AC.



On BC describe (I. 46) the square BDEC, and on BA, AC, the squares GB, HC; and (I. 31) through A draw AL parallel to BD or CE, and join AD, FC; then, because each of the angles

BAC, BAG, is a right angle (Def. 29), the two straight lines AC, AG, are in the same straight line (I. 14); for the same reason, AB and AH are in the same straight line; and because the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is equal to the whole FBC (Ax. 2); and because the two sides AB, BD, are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC, therefore (I. 4) the base AD is equal to the base FC, and the triangle ABD to the triangle FBC. Now, the parallelogram BL is double of the triangle ABD (I. 41), because they are upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC. But the doubles of equals are equal to one other (Ax. 6); therefore the parallelogram BL is equal to the square GB. And, in the same manner, by joining AE, BK, it is demonstrated that the parallelogram CL is equal to the square HC. Therefore the whole square BDEC is equal to the two squares GB, HC; and the square BDEC is described upon the straight line BC, and the squares GB, HC, upon BA, AC; wherefore the square upon the side BC is equal to the squares upon the sides BA, AC.

COR.—The square of either of the sides of a right-angled triangle, is equal to the difference between the squares of the hypotenuse and the other side.

For $BC^2 = AB^2 + AC^2$, and taking AC^2 from both these equals, there remains $BC^2 - AC^2 = AB^2$; or taking AB^2 from both, then $BC^2 - AB^2 = AC^2$.

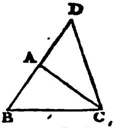
Or, calling the hypotenuse, the base, and the perpendicular, H, B, and P, respectively, $H^2 = B^2 + P^2$. Therefore $H^2 - B^2 = P^2$; and $H^2 - P^2 = B^2$.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

If the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides BA, AC, the angle BAC is a right angle.

From the point A draw (I. 11) AD at right angles to AC, and make AD equal to BA, and join DC. Then, because DA is equal to AB, the square of DA is equal to the square of AB. To each of these add the square of AC; therefore the squares of DA, AC, are equal to the squares of BA, AC. But the square of DC is equal to the squares of DA, AC (I. 47), because DAC is a right angle; and the square of BC, by hypothesis, is equal to the squares of BA, AC; therefore the square of DC is equal to the square of BC; and therefore also the side DC is equal to the side BC. And because the side DA is equal to AB, and AC common to the two triangles DAC, BAC, and the base DC likewise equal to the base BC, the angle DAC is equal to the angle BAC (I. 8). But DAC is a right angle; therefore also BAC is a right angle.



EXERCISES.

1. A line* that bisects the vertical angle of an isosceles triangle, also bisects the base perpendicularly.

2. If the vertical angle of an isosceles triangle be a right angle, each of the angles at the base is half a right angle.

3. If a side of an isosceles triangle be produced beyond the vertex, the exterior angle is double of either of the angles at the base.

4. Of all lines that can be drawn from a given point to a given line, the perpendicular upon it is the least; and of all others, that which is nearest to the perpendicular is less than one more remote; and only two equal lines can be drawn to it from that point, one upon each side of the perpendicular.

5. If the exterior angle and one of the opposite interior angles, in one triangle, be double those of another, the remaining opposite interior angle of the former, is double that of the latter.

Note.—This proposition is the geometrical principle on which the construction of Hadley's quadrant depends.

---

* When the word line is used, straight line is understood, unless otherwise expressed.

6. If two intersecting lines be respectively parallel, or equally inclined, to other two intersecting lines, the inclination of the former is equal to that of the latter.

7. The difference between two sides of a triangle is less than the third side.

8. The sum of two sides of a triangle is greater than twice the line joining the vertex and the middle of the base.

9. Half the base of a triangle is greater than, equal to, or less than, the line joining the middle of the base and the vertex, according as the vertical angle is obtuse, right, or acute.

10. In a given straight line to find a point equally distant from two given points.

11. If two lines bisect the angles at the base of a triangle, the line joining their point of intersection and the vertex, bisects the vertical angle.

12. If two lines bisect perpendicularly two sides of a triangle, the perpendicular, from their point of intersection upon the base, will bisect it.

13. To find a point in a given line such, that lines drawn from it to two given points, will make equal angles with the given line.

14. If a line be bisected perpendicularly by another line, every point in the latter is equally distant from the extremities of the former; and any point not situated in the latter is at unequal distances from the extremities of the former.

15. Through a given point, between two given lines, to draw a line so that the part of it intercepted between them, may be bisected in that point.

16. The angle contained by a line drawn from the vertex of a triangle perpendicular to the base, and another bisecting the vertical angle, is equal to half the difference of the angles at the base.

17. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, their sum will be equal to the perpendicular from either extremity of the base upon the opposite side

18. The sum of the perpendiculars drawn from any point within an equilateral triangle on the sides, is equal to the perpendicular from any of the angular points upon the opposite side.

19. Given the sum of the hypotenuse and a side of a right-angled triangle, and also the remaining side, to construct it.

20. Given the sum of the sides of a triangle and the angles at the base, to construct it.

21. If two lines be drawn from the extremities of the base of a triangle to bisect the opposite sides, the line joining their intersection with the vertex, if produced, will bisect the base.

22. If two polygons be constructed on the same side of the same base, the sum of the sides of the interior polygon, if it be concave internally, is less than the sum of the sides of the exterior figure.

SECOND BOOK.

DEFINITIONS.

1. If a straight line be divided or cut at any point, the point is called a *point of section.*

2. The distance of the point of section from the middle of the line, is called the *mean distance.*

3. If the point of section lie between the extremities of the line, it is said to be cut *internally;* and if beyond one of the extremities, it is said to be cut *externally.*

4. The distances from a point of section of a line to its extremities, are called *segments;* which are said to be *internal* or *external,* according as the line is cut internally or externally.

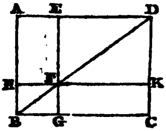
5. The rectangle *under*, or *contained by*, two lines, is a rectangle, of which these lines, or lines equal to them, are two adjacent sides. The rectangle under two lines, as AB and BC, is sometimes concisely expressed thus, $AB \cdot BC$; or if A and B denote the lines, by $A \cdot B$. So the rectangle under a line, which is the sum of two lines A and B, and a third line C, is expressed thus $(A + B)\,C$; that under the excess of A above B, and another line C, by $(A - B)\,C$;

and that under a line equal to A + B, and another equal to A — B, by (A + B) (A — B).

6. When a line is cut into two segments, so that the rectangle under the whole line and one of the segments is equal to the square of the other segment, it is said to be cut *medially*, or in *medial section*.

7. Any of the parallelograms about a diagonal of any parallelogram, together with the two complements, is called a *gnomon*.

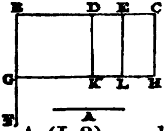
Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines, and let BC be divided into any parts in the points D, E; the rectangle contained by the straight lines A, BC, is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

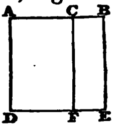


From the point B draw (I. 11) BF at right angles to BC, and make BG equal to A (I. 3); and through G draw GH parallel to BC (I. 31); and through D, E, C, draw (I. 31) DK, EL, CH, parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, because DK, that is, BG (I. 34), is equal to A; and in like manner the rectangle EH is contained by A, EC; therefore the rectangle contained by A, BC, is equal to the several rectangles contained by A, BD, and by A, DE, and also by A, EC.

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle AB · AC, shall be equal to the square of AB.



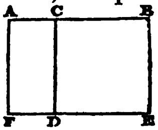
Upon AB describe (I. 46) the square ADEB, and through C draw CF, parallel to AD or BE (I. 31); then AE is equal to the rectangles AF, CE; and AE is the square of AB; and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC, together with the rectangle AB·BC, is equal to the square of AB.

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the foresaid part.

Let the straight line AB be divided into two parts in the point C; the rectangle AB · BC, is equal to the rectangle AC·CB, together with the square of BC.

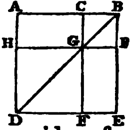
Upon BC describe (I. 46) the square CDEB, and produce ED to F, and through A draw AF parallel to CD or BE (I. 31); then the rectangle AE is equal to the rectangles AD, CE. Now, AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the square of BC; therefore the rectangle AB · BC is equal to the rectangle AC·CB, together with the square of BC.



PROPOSITION IV. THEOREM.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC·CB.



Upon AB describe the square ADEB (I. 46), and join BD, and through C draw CGF parallel to AD or BE (I. 31), and through G draw HK parallel to AB or DE. And because CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal to the interior and opposite angle ADB (I. 29); but ADB is equal to the angle ABD (I. 5), because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle GBC; and therefore the side BC is equal to the side CG (I. 6); but CB is equal also to GK (I. 34), and CG to BK; wherefore the figure CGKB is equilateral. It is likewise rectangular; for the angle CBK being a right angle, the other angles of the parallelogram CGKB are also right angles (I. 46, Cor.); wherefore CGKB is a square, and it is upon the side CB. For the same reason, HF also is a square, and it is upon the side HG, which is equal to AC; therefore HF, CK, are the squares of AC, CB. And because the complement AG is equal (I. 43) to the complement GE, and because AG is the rectangle contained by AC, CG, that is, by AC, CB; GE is also equal to the rectangle AC·CB; wherefore AG, GE, are equal to twice the rectangle AC·CB. And HF, CK, are the squares of AC, CB; wherefore the four figures HF, CK, AG, GE, are equal to the squares of AC, CB, and to twice the rectangle AC·CB. But HF, CK, AG, GE, make up the whole figure ADEB, which is the square of AB; therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC·CB.

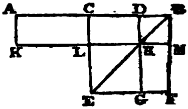
Cor.—From the demonstration, it is manifest that the parallelograms about the diagonal of a square are likewise squares.

PROPOSITION V. THEOREM.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD · DB, together with the square of CD, is equal to the square of CB.

Upon CB describe the square CEFB (I. 46); join BE, and through D draw DHG parallel to CE or BF (I. 31); and through H draw KLM parallel to CB or EF; and also through A draw AK parallel to CL or BM. And because the complement CH is equal to the complement HF (I. 43), to each of these add DM; therefore the whole CM is equal to the whole DF; but CM is equal to AL (I. 36), because AC is equal to CB; therefore also AL is equal to DF. To each of these add CH, and the whole AH is equal to DF and CH; but AH is the rectangle contained by AD, DB, for DH is equal to DB (II. 4, Cor.); and DF, together with CH, is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD · DB. To each of these add LG, which is equal to the square of CD; therefore the gnomon CMG, together with LG, is equal to the rectangle AD · DB, together with the square of CD. But the gnomon CMG and LG make up the whole figures CEFB, which is the square of CB; therefore the rectangle AD · DB, together with the square of CD, is equal to the square of CB.



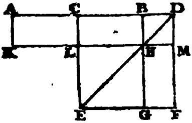
COR.—From this proposition it is manifest, that the difference of the squares of two unequal lines BC, CD, is equal to the rectangle contained by their sum and difference.

For $BC^2 = AD \cdot DB + CD^2$, and taking CD^2 from both these equals, there remains $BC^2 - CD^2 = AD \cdot DB = (BC + CD)(BC - CD)$, for $BC + CD = AC + CD = AD$.

PROPOSITION VI. THEOREM.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the rectangle AD · DB, together with the square of CB, is equal to the square of CD.



Upon CD describe the square CEFD (I. 46); join DE, and through B draw BHG parallel to CE or DF (I. 31), and through H draw KLM parallel to AD or EF, and also through A draw AK parallel to CL or DM. And because AC is equal to CB, the rectangle AL is equal to CH (I. 36); but CH is equal to HF (I. 43); therefore also AL is equal to HF. To each of these add CM; therefore the whole AM is equal to the gnomon CMG. Now, AM is the rectangle contained by AD, DB, for DM is equal to DB (II. 4, Cor.); therefore the gnomon CMG is equal to the rectangle AD · DB; add to each of these LG, which is equal to the square of CB; therefore the rectangle AD · DB, together with the square of CB, is equal to the gnomon CMG, together with LG. But the gnomon CMG, together with LG, makes up the whole figure CEFD, which is the square of CD; therefore the rectangle AD · DB, together with the square of CB, is equal to the square of CD.

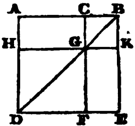
Schol.—The Corollary to Proposition 5 follows in a similar manner from this one. These two propositions are equivalent to the following:—If a line be bisected, and be also cut unequally, either internally or externally, the rectangle under the unequal segments is equal to the difference between the squares of half the line and the mean distance.

PROPOSITION VII. THEOREM.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC, are equal to twice the rectangle AB·BC, together with the square of AC.

Upon AB describe the square ADEB (I. 46), and construct the figure as in the preceding propositions. And because AG is equal to GE (I. 43), add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE, are double of AK. But AK, CE, are the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is double of AK. But twice the rectangle AB · BC is also double of AK, for BK is equal to BC (II. 4, Cor.); therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB·BC. To each of these equals add HF, which is equal to the square of AC; therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB · BC, and the square of AC. Now, the gnomon AKF, together with the squares CK, HF, makes up the whole figure ADEB and CK, which are the squares of AB and BC; therefore the squares of AB and BC are equal to twice the rectangle AB · BC, together with the square of AC.



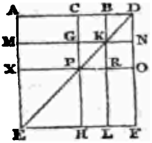
Otherwise: because the square of AB is equal to the squares of AC and CB, together with twice the rectangle AC·CB, if the square of CB be added to both, the squares of AB and CB are equal to the square of AC, together with twice the square of CB, and twice the rectangle AC · CB. But the square of CB, together with the rectangle AC·CB, is equal to the rectangle AB·BC (II. 3); and therefore twice the square of CB, together with twice the rectangle AC·CB, is equal to twice the rectangle AB·BC. Wherefore the squares of AB and CB are equal to twice the rectangle AB·BC, together with the square of AC.

COR.—Hence, the sum of the squares of any two lines is equal to twice the rectangle contained by the lines together with the square of the difference of the lines.

PROPOSITION VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole and the first mentioned part.

Let the straight line AB be divided into any two parts in the point C; four times the rectangle AB · BC, together with the square of AC, is equal to the square of the straight line made up of AB and BC together.



Produce AB to D, so that BD be equal to CB, and upon AD describe the square AEFD; and construct two figures such as in the preceding. Because CB is equal to BD, and that CB is equal to GK (I. 34), and BD to KN; therefore GK is equal to KN. For the same reason, PR is equal to RO; and because CB is equal to BD, and GK to KN, the rectangle CK is equal to BN (I. 36), and GR to RN. But CK is equal to RN (I. 43), because they are the complements of the parallelogram CO; therefore, also, BN is equal to GR; and the four rectangles BN, CK, GR, RN, are therefore equal to one another, and so are quadruple of one of them, CK. Again, because CB is equal to BD, and BD equal to BK (II. 4, Cor.), that is, to CG; and CB equal to GK, that is, to GP; therefore CG is equal to GP; and because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF. But MP is equal to PL (I. 43), because they are the complements of the parallelogram ML; wherefore AG is equal also to RF; therefore the four rectangles AG, MP, PL, RF, are equal to one another, and so are quadruple of one of them, AG. And it was demonstrated, that the four CK, BN, GR, and RN, are quadruple of CK; therefore the eight rectangles which make up the gnomon AOH, are quadruple of AK. And because AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB · BC is quadruple of AK. But the gnomon AOH was demonstrated to be quadruple of AK; therefore four times the rectangle AB · BC is equal to the gnomon AOH. To each of these add XH, which is equal to the square of AC (II. 4, Cor.); therefore four times the rectangle AB·BC, together with the square of AC, is equal to the gnomon AOH and the square XH. But the gnomon AOH and the square XH make up the figure AEFD, which is the square of AD; therefore four times the rectangle AB·BC, together with the square of AC, is equal to the square of AD, that is, of AB and BC added together in one straight line.

Cor. 1.—Hence, because AD is the sum, and AC the difference of the lines AB and BC, four times the rectangle contained by any two lines, together with the square of their difference, is equal to the square of the sum of the lines.

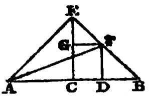
Cor. 2.—From the demonstration, it is manifest, that, since the square of CD is quadruple of the square of CB, the square of any line is quadruple of the square of half that line.

PROPOSITION IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the straight line AB be divided at the point C into two equal, and at D into two unequal parts; the squares of AD, DB, are together double of the squares of AC, CD.

From the point C draw CE at right angles to AB (I. 11), and make it equal to AC or CB, and join EA, EB; through D draw DF parallel to CE (I. 31), and through F draw FG parallel to AB; and join AF. Then, because AC is equal to CE, the angle EAC is equal to the angle AEC (I. 5); and because the angle ACE is a right angle, the two others AEC, EAC, together make one right angle (I. 32); and they are equal to one another; each of them, therefore, is half of a right angle. For the same reason, each of the angles CEB, EBC, is half a right angle; and therefore the whole AEB is a right angle. And because the angle GEF is half a right angle, and EGF a right angle,

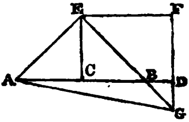


for it is equal to the interior and opposite angle ECB (I. 29), the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the side EG equal to the side GF (I. 6). Again, because the angle at B is half a right angle, and FDB a right angle, for it is equal to the interior and opposite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the side DF to the side DB. Now, because AC is equal to CE, the square of AC is equal to the square of CE; therefore the squares of AC, CE, are double of the square of AC. But the square of EA is equal to the squares of AC, CE, because ACE is a right angle (I. 47); therefore the square of EA is double of the square of AC. Again, because EG is equal to GF, the square of EG is equal to the square of GF; therefore the squares of EG, GF, are double of the square of GF; but the square of EF is equal to the squares of EG, GF; therefore the square of EF is double of the square GF; and GF is equal to CD (I. 34); therefore the square of EF is double of the square of CD. But the square of AE is likewise double of the square of AC; therefore the squares of AE, EF, are double of the squares of AC, CD. And the square of AF is equal to the squares of AE, EF (I. 47), because AEF is a right angle; therefore the square of AF is double of the squares of AC, CD. But the squares of AD, DF, are equal to the square of AF, because the angle ADF is a right angle; therefore the squares of AD, DF, are double of the squares of AC, CD. And DF is equal to DB; therefore the squares of AD, DB, are double of the squares of AC, CD.

PROPOSITION X. THEOREM.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to the point D; the squares of AD, DB, are double of the squares of AC, CD.



From the point C draw CE at right angles to AB (I. 11); and make it equal to AC or CB, and join AE, EB; through E draw EF parallel to AB (I. 31), and through D draw DF parallel to CE. And because the straight line EF meets the parallels EC, FD, the angles CEF, EFD, are equal to two right angles (I. 29); and therefore the angles BEF, EFD, are less than two right angles. But straight lines, which with another straight line make the interior angles upon the same side less than two right angles, do meet if produced far enough (I. 29, Cor.); therefore EB, FD, shall meet, if produced towards B, D; let them meet in G, and join AG. Then, because AC is equal to CE, the angle CEA is equal to the angle EAC (I. 5); and the angle ACE is a right angle; therefore each of the angles CEA, EAC, is half a right angle (I. 32). For the same reason, each of the angles CEB, EBC, is half a right angle; therefore AEB is a right angle. And because EBC is half a right angle, DBG is also half a right angle (I. 15), for they are vertically opposite; but BDG is a right angle, because it is equal to the alternate angle DCE (I. 29); therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG; wherefore also the side DB is equal to the side DG (I. 6). Again, because EGF is half a right angle, and that the angle at F is a right angle, because it is equal to the opposite angle ECD (I. 34), the remaining angle FEG is half a right angle, and equal to the angle EGF; wherefore also the side GF is equal to the side FE. And because EC is equal to CA, the square of EC is equal to the square of CA; therefore the squares of EC, CA, are double of the square of CA. But the square of EA is equal to the squares of EC, CA (I. 47); therefore the square of EA is double of the square of AC. Again, because GF is equal to FE, the square of GF is equal to the square of FE; and therefore the squares of GF, FE, are double of the square of EF. But the square of EG is equal to the squares of GF, FE (I. 47); therefore the square of EG is double of the square of EF; and EF is equal to CD; wherefore the square of EG is double of the square of

CD; but it was demonstrated, that the square of EA is double of the square of AC; therefore the squares of AE, EG, are double of the squares of AC, CD. And the square of AG is equal to the squares of AE, EG; therefore the square of AG is double of the squares of AC, CD. But the squares of AD, DG, are equal to the square of AG; therefore the squares of AD, DG, are double of the squares of AC, CD. But DG is equal to DB; therefore the squares of AD, DB, are double of the squares of AC, CD.

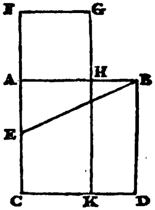
Schol.—The last two propositions are equivalent to the following:—If a straight line be bisected, and be also cut unequally, either internally or externally, the sum of the squares of the two unequal segments is equal to twice the squares of half the line and of the mean distance.

PROPOSITION XI. PROBLEM.

To divide a given straight line in medial section.

Let AB be the given straight line; it is required to divide it in medial section.

Upon AB describe the square ABDC (I. 46); bisect AC in E (I. 10), and join BE; produce CA to F, and make EF equal to EB (I. 3), and upon AF describe the square FGHA; AB is divided in H, so that the rectangle AB·BH is equal to the square of AH.



Produce GH to K; because the straight line AC is bisected in E, and produced to the point F, the rectangle CF, FA, together with the square of AE, is equal to the square of EF (II. 6). But EF is equal to EB; therefore the rectangle CF·FA, together with the square of AE, is equal to the square of EB; and the squares of BA, AE, are equal to the square of EB (I. 47), because the angle EAB is a right angle; therefore the rectangle CF·FA, together with the square of AE, is equal to the squares of BA, AE; take away the square of AE, which is common to both; therefore the remaining rectangle CF·FA is equal to the square of AB. Now, the figure FK is the rectangle contained by CF, FA, for AF is equal to FG, and AD is the square of AB; therefore FK is equal to

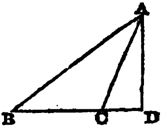
AD. Take away the common part AK, and the remainder FH is equal to the remainder HD. And HD is the rectangle contained by AB, BH, for AB is equal to BD, and FH is the square of AH; therefore the rectangle AB·BH is equal to the square of AH. Wherefore the straight line AB is divided in H so, that the rectangle AB·BH is equal to the square of AH.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced (I. 12); the square of AB is greater than the squares of AC, CB, by twice the rectangle BC·CD.

Because the straight line BD is divided into two parts in the point C, the square of BD is equal to the squares of BC, CD, and twice the rectangle BC·CD (II. 4); to each of these equals add the square of DA; and the squares of DB, DA, are equal to the squares of BC, CD, DA, and twice the rectangle BC·CD. But the square of BA is equal to the squares of BD, DA (I. 47), because the angle at D is a right angle; and the square of CA is equal to the squares of CD, DA. Therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC·CD; that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC·CD.

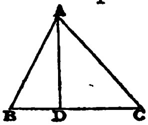


PROPOSITION XIII. THEOREM.

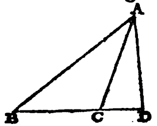
In every triangle, the square of the side subtending any of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle (I. 12); the square of AC, opposite to the angle B, is less than the squares of CB, BA, by twice the rectangle CB·BD.

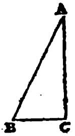
First, let AD fall within the triangle ABC; and because the straight line CB is divided into two parts in the point D, the squares of CB, BD, are equal to twice the rectangle contained by CB, BD, and the square of DC (II. 7). To each of these equals add the square of AD; therefore the squares of CB, BD, DA, are equal to twice the rectangle CB·BD, and the squares of AD, DC. But the square of AB is equal to the squares of BD, DA (I. 47), because the angle BDA is a right angle; and the square of AC is equal to the squares of AD, DC. Therefore the squares of CB, BA, are equal to the square of AC, and twice the rectangle CB·BD; that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB·BD.



Secondly, let AD fall without the triangle ABC; then, because the angle at D is a right angle, the angle ACB is greater than a right angle (I. 16); and therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle BC·CD (II. 12). To these equals add the square of BC, and the squares of AB, BC, are equal to the square of AC, and twice the square of BC, and twice the rectangle BC·CD. But because BD is divided into two parts in C, the rectangle DB·BC is equal to the rectangle BC·CD, and the square of BC (II. 3); and the doubles of these are equal; therefore the squares of AB, BC, are equal to the square of AC, and twice the rectangle DB·BC; therefore the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB·BC.



Lastly, let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares of AB, BC, are equal to the square of AC, and twice the square of BC (I. 47).

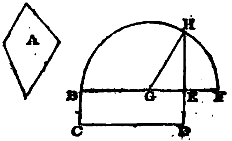


PROPOSITION XIV. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure; it is required to describe a square that shall be equal to A.

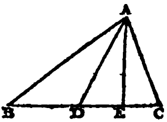
Describe (I. 45) the rectangular parallelogram BCDE equal to the rectilineal figure A. If, then, the sides of it BE, ED, are equal to one another, it is a square, and what was required is done; but if they are not equal, produce one of them BE to F, and make EF equal to ED, and bisect BF in G; and from the centre G, at the distance GB or GF, describe the semicircle BHF, and produce DE to H, and join GH. Therefore, because the straight line BF is divided into two equal parts in the point G, and into two unequal in the point E, the rectangle BE·EF, together with the square of EG, is equal to the square of GF (II. 5). But GF is equal to GH; therefore the rectangle BE·EF, together with the square of EG, is equal to the square of GH. But the squares of HE, EG, are equal to the square of GH (I. 47); therefore, the rectangle BE·EF, together with the square of EG, is equal to the squares of HE, EG. Take away the square of EG, which is common to both, and the remaining rectangle BE·EF is equal to the square of EH; but the rectangle contained by BE·EF is the parallelogram BD, because EF is equal to ED; therefore BD is equal to the square of EH. But BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the square of EH; wherefore a square has been made equal to the given rectilineal figure A; namely, the square described upon EH.



PROPOSITION A. THEOREM.

If one side of a triangle be bisected, the sum of the squares of the other two sides is double of the square of half the side bisected, and of the square of the line drawn from the point of bisection to the opposite angle of the triangle.

Let ABC be a triangle, of which the side BC is bisected in D, and DA drawn to the opposite angle; the squares of BA and AC are together double of the squares of BD and DA.



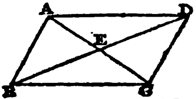
From A draw AE perpendicular to BC, and because BEA is a right angle, the square of AB is equal to the squares of BE and EA (I. 47); for the same reason, the square of AC is equal to the two squares of CE and EA; therefore the squares of BA and AC are equal to the squares of BE and EC, together with twice the square of EA. But because the line BC is cut equally in D, and unequally in E, the squares of BE and EC are equal to twice the squares of BD and DE (II. 9); therefore the squares of BA and AC are equal to twice the square of BD, together with twice the squares of DE and EA. Now, the squares of DE and EA are equal to the square of DA (I. 47), and therefore twice the squares of DE and EA, to twice the square of DA; wherefore, also, the squares of BA and AC are equal to twice the square of BD, together with twice the square of DA.

PROPOSITION B. THEOREM.

The sum of the squares of the diameters of any parallelogram is equal to the sum of the squares of the sides of the parallelogram.

Let ABCD be a parallelogram, of which the diameters are AC and BD; the sum of the squares of AC and BD is equal to the sum of the squares of AB, BC, CD, DA.

Let AC and BD intersect one another in E; and because the vertical angles AED, CEB, are equal (I. 15), and also the alternate angles EAD, ECB (I. 29), the triangles ADE,



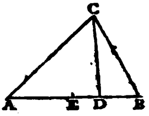
CEB, have two angles in the one equal to two angles in the other, each to each; but the sides AD and BC, which are opposite to equal angles in these triangles, are also equal (I. 34); therefore the other sides which are opposite to the equal angles are equal (I. 26); namely, AE to EC, and ED to EB.

Since, therefore, BD is bisected in E, the squares of BA and AD are equal to twice the square of BE, together with twice the square of EA (II. A); and for the same reason the squares of BC and CD are equal to twice the square of BE, together with twice the square of EC, that is, of EA, because EC is equal to EA; therefore the four squares of BA, AD, DC, CB, are equal to four times the squares of BE and of EA. But the square of BD is equal to four times the square of BE, because BD is double of BE (II. 8, Cor.); and for the same reason, the square of AC is equal to four times the square of AE; wherefore, also, the squares of BD and AC are equal to the four squares of BA, AD, DC, CB.

Cor.—From the demonstration, it is manifest that the diameters of every parallelogram bisect one another.

PROPOSITION C. THEOREM.

The difference between the squares of the two sides of a triangle, is equal to the difference between the squares of the segments of the base, formed by a perpendicular upon it from the vertex.



Let ABC be a triangle, and CD a perpendicular on the base AB, then $AC^2 - BC^2 = AD^2 - DB^2$.

For (I. 47) $AC^2 = AD^2 + DC^2$ and $BC^2 = DB^2 + DC^2$; therefore the difference between AC^2 and BC^2 is equal to that between $AD^2 + DC^2$ and $DB^2 + DC^2$; or $AC^2 - BC^2 = AD^2 - DB^2$.

Cor. 1.—The rectangle under the sum and difference of the two sides of a triangle, is equal to twice the rectangle under the base and the distance of its middle point from the perpendicular on it from the vertex.

Let AB be bisected in E, then $(AC + CB)(AC - CB) = 2\ AB \cdot ED$.

For $AC^2 - CB^2 = AD^2 - DB^2$; therefore (II. 5, Cor.) $(AC + CB)(AC - CB) = (AD + DB)(AD - DB) = AB \cdot 2\ ED = 2\ AB \cdot ED$; because $AD = AE + ED = BE + ED = DB + 2\ ED$, therefore $AD - DB = 2\ ED$, and the rectangle under AB and 2 ED, is equal to twice that under AB and ED.

Cor. 2.—If from any point in a given line a perpendicular be drawn to it, and, from any points in the latter, lines be drawn to the extremities of the former, the differences between the squares of every two lines from the same point are equal.

For these differences are all equal to the difference between the squares of the segments of the given line.

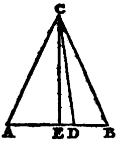
Cor. 3.—If the difference between the squares of two lines joining a point and the extremities of a given line be equal to that of other two lines similarly drawn from another point, the line joining these two points is perpendicular to the given line, if the points be on the same side of a line bisecting the given line perpendicularly.

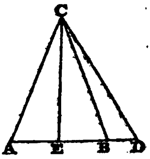
For the segments of the given line, formed by perpendiculars upon it from these two points, must be equal (Cor. 2), or the two perpendiculars must coincide.

PROPOSITION D. THEOREM.

If from the vertex of an isosceles triangle a line be drawn cutting the base, either internally or externally, the difference between the squares of this line and either side is equal to the rectangle under the segments of the base.

Let ABC be an isosceles triangle, and CD a line from the vertex cutting the base, then (1st figure) $AC^2 - CD^2 = AD \cdot DB$, and (2d fig.) $CD^2 - AC^2 = AD \cdot DB$.





For (1st fig.) $AC^2 - CD^2 = AE^2 - ED^2$, (II. c.), $= (AE + ED)(AE - ED)$ (II. 5, Cor.) $= AD \cdot DB$, for $AE - ED = BE - ED = DB$.

And (2d fig.), $CD^2 - AC^2 = ED^2 - AE^2$, (II. c.), $=$

(ED + AE) (ED — AE) (II. 5, Cor.) = AD·DB, for ED — AE = ED — EB = DB.

Cor. 1.—*If + refer to the 1st figure, and — to the 2d,* $AC^2 = CD^2 \pm AD \cdot DB$.

For, in the 1st figure, $AC^2 - CD^2 = AD \cdot DB$; therefore, adding to these equals CD^2, $AC^2 = CD^2 + AD \cdot DB$. And, in 2d fig. $CD^2 - AC^2 = AD \cdot DB$, and, adding to these equals AC^2, and taking from them AD·DB; therefore $AC^2 = CD^2 - AD \cdot DB$.

Cor. 2.—*Half the difference of two unequal lines added to half their sum, gives the greater; and taken from half the sum, gives the less.*

For AD, DB (1st fig.) are two unequal lines, AE is half their sum, and ED half their difference; and AE + ED = AD, and AE — ED = DB.

EXERCISES.

1. The square of one of the sides of a right-angled triangle is equal to the rectangle under the sum and difference of the hypotenuse and the other side.

2. The square of a perpendicular upon the hypotenuse of a right-angled triangle drawn from the opposite angle, is equal to the rectangle under the segments of the hypotenuse.

3. If a line be cut in medial section, the line composed of it and its greater segment is similarly divided.

4. If from any point lines be drawn to the angular points of a rectangle, the sums of the squares of those drawn to opposite angles are equal.

5. The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of its diagonals, and four times the square of the line joining their middle points.

6. The sum of the squares of two opposite sides of a quadrilateral, together with four times the square of the line joining their middle points, is equal to the sum of the squares of the other two sides and of the diagonals.

7. To bisect a given triangle by a line drawn from a point in one of its sides.

8. The squares of the sum, and of the difference of two lines, are together double the squares of these lines.

9. To bisect a parallelogram by a line drawn from a point in one of its sides.

10. A line joining the middle points of two sides of a triangle, is parallel to the base, and equal to the half of it.

11. The quadrilateral formed by joining the successive middle points of the sides of a given quadrilateral, is a parallelogram.

12. The sum of the squares of the diagonals of a quadrilateral, is equal to twice the sum of the squares of the lines joining the middle points of the opposite sides.

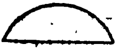
THIRD BOOK.

DEFINITIONS.

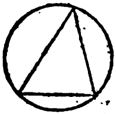
1. An *arc* of a circle is any portion of the circumference.

2. The *chord* of an arc is a straight line joining its extremities.

3. The arc of a semicircle is called a *semicircumference*; and a radius is called a *semidiameter*.

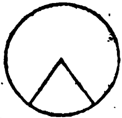


4. A *segment* of a circle is a figure contained by an arc and its chord.

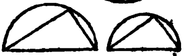


5. An *angle in a segment* is an angle contained by two lines, drawn from any point in its arc, to the extremities of its chord.

6. An angle is said to *insist* or *stand* upon the arc which subtends it.



7. A *sector* of a circle is a figure contained by two radii and the intercepted arc. When the radii are perpendicular, the sector is called a *quadrant*.



8. *Similar segments* of a circle are those that contain equal angles.

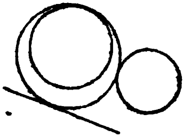
9. *Similar arcs* of circles are those that subtend equal angles at the centre.

10. *Similar sectors* are those that are bounded by similar arcs.

11. *Equal circles* are those that have equal radii; and *concentric* circles are those that have the same centre.

12. A straight line is said to *touch* a circle, or to be a *tangent* to it, when it meets the circle, and being produced does not cut it.

13. Circles are said to *touch* one another when they meet, but do not cut one another. Such circles may be called *tangent* circles.



14. The point in which a tangent and a circle, or two tangent circles, meet, is called the *point of contact.*

15. Chords are said to be *equally distant* from the centre of a circle, when the perpendiculars upon them, from the centre, are equal; and the chord on which the greater perpendicular falls, is said to be *farther* from the centre.

16. A *secant* is a straight line drawn from a point, without a circle, and meeting it in two points. A secant may be considered to be a chord produced, the point without the circle from which it is drawn being a point of external section.

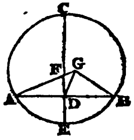
PROPOSITION I. PROBLEM.

To find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre.

Draw within it any chord AB, and bisect it in D (I. 10); from the point D draw DC at right angles to AB (I. 11), and produce it to E, and bisect CE in F. The point F is the centre of the circle ABC.

For, if it be not, let, if possible, G be the centre, and join GA, GD, GB. Then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD, DG, are equal to the two BD, DG, each to each; and the radius GA is equal to the radius GB; therefore the angle ADG is equal to the angle GDB (I. 8); therefore (I. Def. 10) the angle GDB is a right angle. But FDB is likewise a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the less, which is impossible; therefore G is not the centre of the circle ABC. In the same manner it can be shown, that no



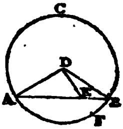
other point but F is the centre; that is, F is the centre of the circle ABC.

Cor.—From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROPOSITION II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A, B, any two points in the circumference; the straight line drawn from A to B shall fall within the circle.



Take any point in AB as E; find D the centre of the circle ABC; join AD, DB, and DE, and let DE meet the circumference in F. Then, because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5); and because AE, a side of the triangle DAE, is produced to B, the angle DEB is greater than the angle DAE (I. 16); but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE; DB is therefore greater than DE (I. 19). But DB is equal to DF; wherefore DF is greater than DE, and the point E is therefore within the circle. The same may be demonstrated of any other point between A and B; therefore AB is within the circle.

Schol.—When the two points are at the extremities of a diameter, or diametrically opposite, the truth of the proposition is evident. It is obvious, from the nature of the circle, when they are at a less distance. When the points are exceedingly near, however, it is not then so evident that the chord lies wholly within the circle, and no part of it on the arc; and therefore, in this case, it is necessary to demonstrate the proposition. Some parts of the figure, however, in this extreme case, would be so small as to be indistinct, and therefore it is necessary to assume the points at a sufficient distance; and as the reasoning in this case applies to every case, the proposition will therefore be established also in the extreme case. A similar observation applies to se-

Educ [illegible]

[illegible]

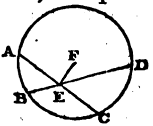
[illegible]

, and of bisecting a chord—if a line
lfil the third.

TION IV. THEOREM.

rds cut one another which do not
ntre, they do not bisect each other.

le, and AC, BD, two chords in it,
n the point E, and do not both pass
, BD, do not bisect one another:
et AE be equal to EC, and BE to
pass through the centre, it is plain
by the other which
he centre. But if
rough the centre,
circle, and join EF

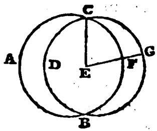


E, a straight line
s another AC which does not pass
all cut it at right angles (III. 3);
angle. Again, because the straight
ht line BD, which does not pass
ll cut it at right angles; wherefore
nd FEA was shown to be a right
equal to the angle FEB, the less
impossible; therefore AC, BD, do

TION V. THEOREM.

another, they shall not have the

BC, CDG, cut one another in the
not the same centre.
let E be their centre; join EC, and
EFG meeting them
use E is the centre of
equal to EF. Again,
re of the circle CDG,
but CE was shown to
erefore EF is equal to



reater, which is impossible; therefore
f the circles ABC, CDG.

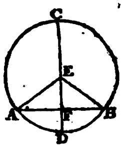
veral propositions, which appear to be axiomatic, except in the extreme case, as I. 20.

PROPOSITION III. THEOREM.

If a straight line drawn through the centre of a circle bisect a chord in it which does not pass through the centre, it shall cut it at right angles; and if it cuts it at right angles, it shall bisect it.

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any chord AB, which does not pass through the centre, in the point F; it cuts it also at right angles.

Take E the centre of the circle (III. 1), and join EA, EB. Then, because AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two sides in the one equal to two sides in the other, and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE (I. 8); therefore (I. Def. 10) each of the angles AFE, BFE, is a right angle; wherefore the straight line CD, drawn through the centre bisecting another AB that does not pass through the centre, cuts the same at right angles.



But let CD cut AB at right angles; CD also bisects it, that is, AF is equal to FB.

The same construction being made, because EA, EB, from the centre are equal to one another, the angle EAF is equal to the angle EBF (I. 5); and the right angle AFE is equal to the right angle BFE; therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other, and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (I. 26); AF therefore is equal to FB.

Cor.—If a line bisect a chord of a circle and be perpendicular to it, it will pass through the centre.

For a line from the centre bisecting the chord is perpendicular to it, and therefore coincides with the former perpendicular; therefore it must also pass through the centre.

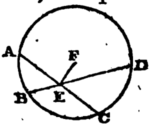
Schol.—It appears, therefore, that of the three conditions—of passing through the centre of a circle, of being perpendicular to a chord, and of bisecting a chord—if a line fulfil two, it will also fulfil the third.

PROPOSITION IV. THEOREM.

If in a circle two chords cut one another which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD, two chords in it, which cut one another in the point E, and do not both pass through the centre; AC, BD, do not bisect one another.

For, if it is possible, let AE be equal to EC, and BE to ED. If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre. But if neither of them pass through the centre, take F the centre of the circle, and join EF (III. 1); and because FE, a straight line through the centre, bisects another AC which does not pass through the centre, it shall cut it at right angles (III. 3); wherefore FEA is a right angle. Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it shall cut it at right angles; wherefore FEB is a right angle; and FEA was shown to be a right angle; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible; therefore AC, BD, do not bisect one another.

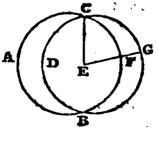


PROPOSITION V. THEOREM.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG, cut one another in the points B, C; they have not the same centre.

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG meeting them in F and G; and because E is the centre of the circle ABC, CE is equal to EF. Again, because E is the centre of the circle CDG, CE is equal to EG; but CE was shown to be equal to EF; therefore EF is equal to EG, the less to the greater, which is impossible; therefore E is not the centre of the circles ABC, CDG.

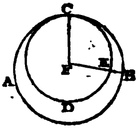


PROPOSITION VI. THEOREM.

If two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE, touch one another internally in the point C; they have not the same centre.

For, if they can, let it be F; join FC, and draw any straight line FEB meeting them in E and B; and because F is the centre of the circle ABC, CF is equal to FB; also, because F is the centre of the circle CDE, CF is equal to FE; and CF was shown equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible; wherefore F is not the centre of the circles ABC, CDE.

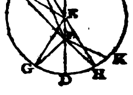


PROPOSITION VII. THEOREM.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: And from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the shortest line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre. Let the centre be E; of all the straight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greatest, and FD, the other part of the diameter AD, is the least; and of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE; and (I. 20) BE, EF, are greater than BF; but AE is equal to EB; therefore AE, EF, that is, AF is greater than BF. Again, because BE is equal to CE, and FE common to the triangles BEF, CEF, the two sides BE, EF, are equal to the two CE, EF; but the angle BEF is greater than the angle CEF; therefore the base BF is greater than the base FC (I. 24);



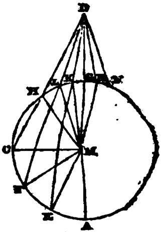
for the same reason, CF is greater than GF. Again, because GF, FE, are greater than EG, and EG is equal to ED; GF, FE, are greater than ED; take away the common part FE, and the remainder GF is greater than the remainder FD; therefore FA is the greatest, and FD the least of all the straight lines from F to the circumference; and BF is greater than CF, and CF than GF.

Also, there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD; at the point E in the straight line EF, make the angle FEH equal to the angle GEF (I. 23), and join FH. Then, because GE is equal to EH, and EF common to the two triangles GEF, HEF; the two sides GE, EF, are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; therefore the base FG is equal to the base FH (I. 4). But besides FH, no straight line can be drawn from F to the circumference equal to FG; for, if there can, let it be FK; and because FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which passes through the centre, is equal to one which is more remote, which is impossible.

PROPOSITION VIII. THEOREM.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than the more remote; but of those which fall upon the convex circumference, the least is that between the point without the circle, and the diameter; and of the rest, that which is nearer to the least is always less than the more remote; and only two equal straight lines can be drawn from the point to the circumference, one upon each side of the least.

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC, be drawn to the circumference, whereof DA passes through the centre. Of those which fall upon the concave part of the circumference AEFC, the greatest is AD which passes through the centre; and the nearer to it is always greater than the more remote; namely, DE than DF, and DF than DC; but of those which fall upon the convex circumference HLKG, the least is DG between the point D and the diameter AG; and the nearer to it is always less than the more remote; namely, DK than DL, and DL than DH.



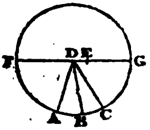
Take M the centre of the circle ABC (III. 1), and join ME, MF, MC, MK, ML, MH; and because AM is equal to ME, add MD to each; therefore AD is equal to EM, MD; but EM, MD, are greater then ED (I. 20); therefore, also, AD is greater than ED. Again, because ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD, are equal to FM, MD; but the angle EMD is greater than the angle FMD; therefore the base ED is greater than the base FD (I. 24). In like manner it may be shown that FD is greater than CD; therefore DA is the greatest, and DE greater than DF, and DF than DC. And because MK, KD, are greater than MD (I. 20), and MK is equal to MG, the remainder KD is greater than the remainder GD (Ax. 5), that is, GD is less than KD. And because MK, DK, are drawn to the point K within the triangle MLD from M, D, the extremities of its side MD; MK, KD, are less than ML, LD (I. 21), whereof MK is equal to ML; therefore the remainder DK is less than the remainder DL. In like manner it may be shown that DL is less than DH; therefore DG is the least, and DK less than DL, and DL than DH. Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least; at the point M, in the straight line MD, make the angle DMB equal to the angle DMK, and join DB; and, because in the triangles KMD, BMD, the side KM is equal to the side BM, and MD common to both, and also the angle KMD equal to the angle BMD, the base DK is equal to the base DB (I. 4). But, besides DB, no straight line can be drawn from D to the circumference, equal to DK; for, if there can, let it be DN; then, because DN is equal to DK, and DK equal to DB, DB is equal to DN; that is, the nearer to DG, the least, equal to the more remote, which has been shown to be impossible.

PROPOSITION IX. PROBLEM.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, namely, DA, DB, DC, the point D is the centre of the circle.

For, if not, let E be the centre; join DE, and produce it to the circumference in F, G; then FG is a diameter of the circle ABC. And because in FG, the diameter of the circle ABC, there is taken the point D which is not the centre, DG shall be the greatest line from it to the circumference, and DC greater than DB (III. 7), and DB than DA; but they are likewise equal, which is impossible; therefore E is not the centre of the circle ABC. In like manner it may be demonstrated that no other point but D is the centre; D therefore is the centre.

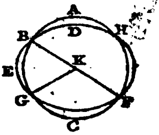


Schol.—The proposition is easily proved without a figure, thus:—The point from which more than two equal lines can be drawn to the circumference of a circle, cannot be a point which is not the centre (III. 7); therefore it must be the centre.

PROPOSITION X. THEOREM.

The circumference of a circle cannot cut that of another in more than two points.

If it be possible, let the circumference FAB cut the circumference DEF in more than two points; namely, in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF; and because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, the point K is the centre of the circle DEF



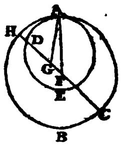
(III. 9); but K is also the centre of the circle ABC; therefore the same point is the centre of two circles that cut one another, which is impossible (III. 5).

PROPOSITION XI. THEOREM.

If two circles touch each other internally, the straight line which joins their centres being produced, shall pass through the point of contact.

Let the two circles ABC, ADE, touch each other internally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE; the straight line which joins the centres F, G, being produced, passes through the point A.

For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG; and because AG, GF, are greater than FA (I. 20), that is, than FH, for FA is equal to FH, being each a radius of the same circle; take away the common part FG, and the remainder AG will be greater than the remainder GH. But AG is equal to GD, therefore GD is greater than GH; and it is also less, which is impossible; therefore the straight line which joins the points F and G cannot fall otherwise than on the point A; that is, it must pass through A.

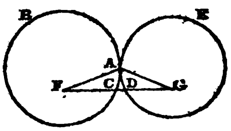


PROPOSITION XII. THEOREM.

If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE, touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE; the straight line which joins the points F, G, shall pass through the point of contact A.

For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG; and because F is the centre of the circle ABC, AF is equal to FC. Also, because G is the centre of the circle ADE, AG is equal to GD; therefore FA, AG, are equal to FC, DG; wherefore the whole FG is greater

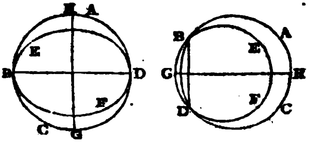


than FA, AG; but it is also less (I. 20), which is impossible; therefore the straight line which joins the points F, G, shall not pass otherwise than through the point of contact A; that is, it must pass through it.

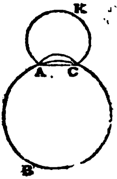
PROPOSITION XIII. THEOREM.

One circle cannot touch another in more points than one, whether it touches it internally or externally.

For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first internally, in the points B, D; join BD, and draw GH bisecting BD at right angles (I. 10, 11); therefore, because the points B, D, are in the circumference of each of the circles, the straight line BD falls within each of them (III. 2); and their centres are in the straight line GH which bisects BD at right angles (III. 1, Cor.); therefore GH passes through the point of contact (III. 11); but it does not pass through it, because the points B, D, are without the straight line GH, which is absurd; therefore one circle cannot touch another on the inside in more points than one.



Nor can two circles touch one another externally in more than one point; for, if it be possible, let the circle ACK touch the circle ABC in the points A, C, and join AC; therefore, because the two points A, C, are in the circumference of the circle ACK, the straight line AC which joins them shall fall within the circle ACK (III. 2). And the circle ACK is without the circle ABC; and therefore the straight line AC is without this last circle; but because the points A, C, are in the circumference of the circle ABC, the straight line AC must be within the same circle, which is absurd; therefore one circle cannot touch another on the outside in more than one point; and it has been shown that they cannot touch on the inside in more points than one.

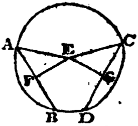


PROPOSITION XIV. THEOREM.

Equal chords in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

Let the chords AB, CD, in the circle ABDC, be equal to one another; they are equally distant from the centre.

Take E the centre of the circle ABDC, and from it draw EF, EG, perpendiculars to AB, CD; then, because the straight line EF, passing through the centre, cuts the chord AB, which does not pass through the centre, at right angles, it also bisects it (III. 3); wherefore AF is equal to FB, and AB double of AF. For the same reason, CD is double of CG; and AB is equal to CD; therefore AF is equal to CG. And because AE is equal to EC, the square of AE is equal to the square of EC. But the squares of AF, FE, are equal to the square of AE (I. 47), because the angle AFE is a right angle; and, for the like reason, the squares of EG, GC, are equal to the square of EC; therefore the squares of AF, FE, are equal to the squares of CG, GE, of which the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square of FE is equal to the remaining square of EG, and the straight line EF is therefore equal to EG; therefore (III. Def. 15) AB, CD, are equally distant from the centre.



Next, if the chords AB, CD, be equally distant from the centre, that is, if FE be equal to EG, AB is equal to CD. For, the same construction being made, it may, as before, be demonstrated, that AB is double of AF, and CD double of CG, and that the squares of EF, FA, are equal to the squares of EG, GC; of which the square of FE is equal to the square of EG, because FE is equal to EG; therefore the remaining square of AF is equal to the remaining square of CG; and the straight line AF is therefore equal to CG. And AB is double of AF, and CD double of CG; wherefore AB is equal to CD.

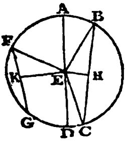
Schol.—A principle employed in this and the next proposition is—if two quantities A and B be together equal to other two C and D together, that is, if $A + B = C + D$; then, if $A = C$, $B = D$; or, if $A > C$, $B < D$.

PROPOSITION XV. THEOREM.

The diameter is the greatest chord in a circle; and of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD is greater than any chord BC which is not a diameter, and BC greater than FG.

From the centre draw EH, EK, perpendiculars to BC, FG, and join EB, EC, EF; and because AE is equal to EB, and ED to EC, AD is equal to EB, EC. But EB, EC, are greater than BC (I. 20); wherefore, also, AD is greater than BC.



And because BC is nearer to the centre than FG, EH is less than EK (III. Def. 15). But, as was demonstrated in the preceding, BC is double of BH, and FG double of FK, and the squares of EH, HB, are equal to the squares of EK, KF, of which the square of EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK, and the straight line BH greater than FK; and therefore BC is greater than FG.

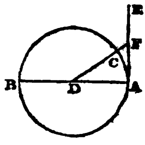
Next, let BC be greater than FG; BC is nearer to the centre than FG; that is, the same construction being made, EH is less than EK; because BC is greater than FG, BH likewise is greater than KF; and the squares of BH, HE, are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK.

PROPOSITION XVI. THEOREM.

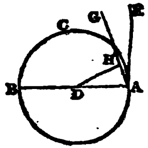
The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn between that straight line and the circumference from the extremity, so as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB; and let AE be drawn from A perpendicular to AB, AE shall fall without the circle.

In AE take any point F; join DF, and let DF meet the circle in C. Because DAF is a right angle, it is greater than the angle AFD (I. 32); therefore DF is greater than DA (I. 19); and DA is equal to DC; therefore DF is greater than DC, and the point F is therefore without the circle. Now, F is any point whatever in the line AE; therefore AE falls without the circle.



Again, between the straight line AE and the circumference no straight line can be drawn from the point A, which does not cut the circle. Let AG be drawn, making the angle DAG less than a right angle; from D draw DH at right angles to AG; and because the angle DHA is a right angle, each of the other angles of the triangle DAH is less than a right angle; the angle DAH is therefore less than the angle DHA, and therefore also the side DH is less than the side DA. The point H, therefore, is within the circle, and therefore the straight line AG cuts the circle.



COR.—From this it is manifest that the straight line which is drawn at right angles to the radius of a circle from the extremity of it, is a tangent; and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it (III. 2). Also, it is evident that there can be but one straight line which touches the circle in the same point.

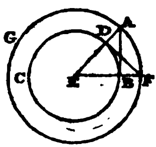
PROPOSITION XVII. PROBLEM.

To draw a straight line from a given point, either without or in the circumference, which shall be a tangent to a given circle.

First, let A be a given point without the given circle BCD;

it is required to draw a straight line from A which shall be a tangent to the circle.

Find the centre E of the circle (III. 1), and join AE; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw DF at right angles to EA (I. 11), and join EF, AB. AB touches the circle BCD.



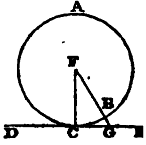
Because E is the centre of the circles BCD, AFG; EA is equal to EF, and ED to EB; therefore the two sides AE, EB, are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB, and the triangle FED to the triangle AEB, and the other angles to the other angles (I. 4); therefore the angle EBA is equal to the angle EDF. But EDF is a right angle, wherefore EBA is a right angle; and EB is a radius; but a straight line drawn from the extremity of a radius, at right angles to it, is a tangent (III. 16, Cor.); therefore AB touches the circle, and it is drawn from the given point A.

But if the given point be in the circumference of the circle, as the point D, draw the radius DE and DF at right angles to DE; DF touches the circle (III. 16, Cor.)

PROPOSITION XVIII. THEOREM.

If a straight line be a tangent to a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the tangent.

Let the straight line DE be a tangent to the circle ABC in the point C; take the centre F, and draw the straight line FC; FC is perpendicular to DE.



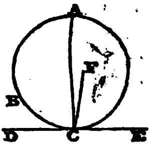
For, if it be not, from the point F draw FBG perpendicular to DE; and because FGC is a right angle, GCF is an acute angle (I. 17); therefore (I. 19) FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the less than the greater, which is impossible; wherefore FG is not perpendicular to DE. In the same manner it may be shown, that no other is perpendicular to it besides FC; that is, FC is perpendicular to DE.

PROPOSITION XIX. THEOREM.

If a straight line be a tangent to a circle, and from the point of contact a straight line be drawn perpendicular to the tangent, the centre of the circle shall be in that line.

Let the straight line DE be a tangent to the circle ABC, in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.

For, if not, let F be the centre, if possible, and join CF. Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE (III. 18); therefore FCE is a right angle. But ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible; wherefore F is not the centre of the circle ABC. In the same manner, it may be shown, that no other point which is not in CA, is the centre; that is, the centre is in CA.



Cor.—If a line, drawn from the centre of a circle, be perpendicular to a tangent, it will pass through the point of contact.

For the line drawn from the centre to the point of contact is perpendicular to the tangent, and it must therefore coincide with the former line, which therefore must pass through the point of contact.

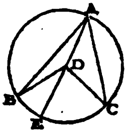
Schol.—From the last two propositions and corollary, it appears that, of the three conditions—of passing through the centre of a circle, of being perpendicular to a tangent, and of passing through the point of contact—if a line fulfil two, it also fulfils the third.

PROPOSITION XX. THEOREM.

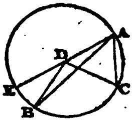
The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same arc.

Let ABC be a circle, and BDC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base; the angle BDC is double of the angle BAC.

First, let D, the centre of the circle, be within the angle BAC, and join AD, and produce it to E. Because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5); therefore the angles DAB, DBA, are double of the angle DAB; but (I. 32) the angle BDE is equal to the angles DAB, DBA; therefore also the angle BDE is double of the angle DAB. For the same reason, the angle EDC is double of the angle DAC; therefore the whole angle BDC is double of the whole angle BAC.



Again, let D, the centre of the circle, be without the angle BAC, and join AD and produce it to E. It may be demonstrated, as in the first case, that the angle EDC is double of the angle DAC, and that EDB, a part of the first, is double of DAB, a part of the other; therefore the remaining angle BDC is double of the remaining angle BAC.

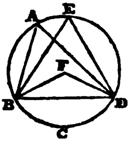


PROPOSITION XXI. THEOREM.

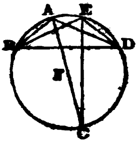
The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED, angles in the same segment BAED; the angles BAD, BED, are equal to one another.

Take F the centre of the circle ABCD; and, first, let the segment BAED be greater than a semicircle, and join BF, FD. And because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same arc, namely, BCD, for their base; therefore the angle BFD is double of the angle BAD. For the same reason, the angle BFD is double of the angle BED; therefore the angle BAD is equal to the angle BED.



But if the segment BAED be not greater than a semicircle, let BAD, BED, be angles in it; these also are equal to one another. Draw AF to the centre, and produce it to C, and join CE; therefore the segment BADC is greater than a semicircle; and the angles in it BAC, BEC, are equal, by the first case. For the same reason, because CBED is greater than a semicircle, the angles CAD, CED, are equal; therefore the whole angle BAD is equal to the whole angle BED.

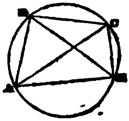


PROPOSITION XXII. THEOREM.

The opposite angles of any quadrilateral figure described in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its opposite angles are together equal to two right angles.

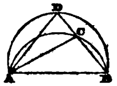
Join AC, BD; and because the three angles of every triangle are equal to two right angles (I. 32), the three angles of the triangle CAB, namely, the angles CAB, ABC, BCA, are equal to two right angles. But the angle CAB is equal to the angle CDB (III. 21), because they are in the same segment BADC, and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB; therefore the whole angle ADC is equal to the angles CAB, ACB. To each of these equals add the angle ABC; therefore the angles ABC, CAB, BCA, are equal to the angles ABC, ADC. But ABC, CAB, BCA, are equal to two right angles; therefore also the angles ABC, ADC, are equal to two right angles. In the same manner, the angles BAD, DCB, may be shown to be equal to two right angles.



PROPOSITION XXIII. THEOREM.

Upon the same straight line and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

If it be possible, let the two similar segments of circles, namely, ACB, ADB, be upon the same side of the same straight line AB, not coinciding with one another; then,

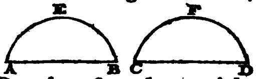


because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (III. 10); one of the segments must therefore fall within the other; let ACB fall within ADB, and draw the straight line BCD, and join CA, DA; and because the segment ACB is similar to the segment ADB, the angle ACB is equal to the angle ADB (III. Def. 8), the exterior to the interior, which is impossible (I. 16).

PROPOSITION XXIV. THEOREM.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD, be similar segments of circles upón the equal straight lines AB, CD; the segment AEB is equal to the segment CFD.

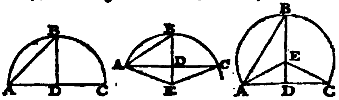


For, if the segment AEB be applied to the segment CFD, so as the point A be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD; therefore the straight line AB coinciding with CD, the segment AEB must coincide with the segment CFD (III. 23), and therefore is equal to it.

PROPOSITION XXV. PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment of a circle; it is required to describe the circle of which it is the segment.



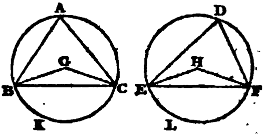
Bisect AC in D (I. 10), and from the point D draw DB at right angles to AC (I. 11), and join AB; first, let the angles ABD, BAD, be equal to one another; then the straight line BD is equal to DA (I. 6), and therefore to DC; and because the three straight lines DA, DB, DC, are all equal, D is the centre of the circle (III. 9); from the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this shall pass through the other points; and the circle, of which ABC is a segment, is described; and because the centre D is in AC, the segment ABC is a semicircle; but if the angles ABD, BAD, are not equal to one another, at the point A, in the straight line AB, make the angle BAE equal to the angle ABD (I. 23), and produce BD, if necessary, to E, and join EC; and because the angle ABE is equal to the angle BAE, the straight line BE is equal to EA; and because AD is equal to DC, and DE common to the triangles ADE, CDE, the two sides AD, DE, are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the base AE is equal to the base EC (I. 4); but AE was shown to be equal to EB, wherefore also BE is equal to EC; and the three straight lines AE, EB, EC, are therefore equal to one another; wherefore E is the centre of the circle (III. 9). From the centre E, at the distance of any of the three AE, EB, EC, describe a circle, this shall pass through the other points; and the circle, of which ABC is a segment, is described; and it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle; but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle; wherefore a segment of a circle being given, the circle is described of which it is a segment.

PROPOSITION XXVI. THEOREM.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Let ABC, DEF, be equal circles, and the equal angles BGC, EHF, at their centres, and BAC, EDF, at their circumferences; the arc BKC is equal to the arc ELF.

Join BC, EF; and because the circles ABC, DEF, are equal, the straight lines drawn from their centres are equal; therefore the two sides BG, GC, are equal to the two EH, HF; and the angle at G is equal to the angle at H; therefore the base BC is equal to the base EF (I. 4); and

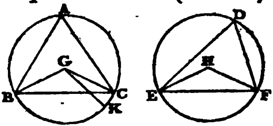


because the angle at A is equal to the angle at D, the segment BAC is similar to the segment EDF (III. Def. 8); and they are upon equal straight lines BC, EF; but similar segments of circles upon equal straight lines are equal to one another (III. 24); therefore the segment BAC is equal to the segment EDF; but the whole circle ABC is equal to the whole DEF; therefore the remaining segment BKC is equal to the remaining segment ELF, and the arc BKC to the arc ELF.

PROPOSITION XXVII. THEOREM.

In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF, at the centres, and BAC, EDF, at the circumferences of the equal circles ABC, DEF, stand upon the equal arcs BC, EF; the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.



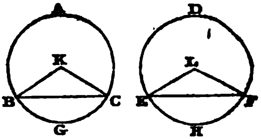
If the angle BGC be equal to the angle EHF, it is manifest that the angle BAC is also equal to EDF (III. 20). But, if not, one of them is the greater; let BGC be the greater, and at the point G, in the straight line BG, make the angle BGK equal to the angle EHF (I. 23); but equal angles stand upon equal arcs (III. 26), when they are at the centre; therefore the arc BK is equal to the arc EF; but EF is equal to BC; therefore also BK is equal to BC, the less to the greater, which is impossible; therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it; and the angle at A is half of the angle BGC, and the angle at D half of the angle EHF; therefore the angle at A is equal to the angle at D.

PROPOSITION XXVIII. THEOREM.

In equal circles, equal chords cut off equal arcs, the greater equal to the greater, and the less to the less.

Let ABC, DEF, be equal circles, and BC, EF, equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less BGC, EHF; the greater BAC is equal to the greater EDF; and the less BGC to the less EHF.

Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF; and because the circles are equal, their radii are equal; therefore BK, KC, are equal to EL, LF; and the base BC is equal to the base EF; therefore the angle BKC is equal to the angle ELF (I. 8). But equal angles stand upon equal arcs (III. 26), when they are at the centres; therefore the arc BGC is equal to the arc EHF. But the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, namely, BAC, is equal to the remaining part EDF.

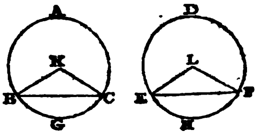


PROPOSITION XXIX. THEOREM.

In equal circles equal arcs are subtended by equal chords.

Let ABC, DEF, be equal circles, and let the arcs BGC, EHF, also be equal, and join BC, EF; the chord BC is equal to the chord GF.

Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF; and because the arc BGC is equal to the arc EHF, the angle BKC is equal to the angle ELF (III. 27); and because the circles ABC, DEF, are equal, their radii are equal; therefore BK, KC, are equal to EL, LF, and they contain equal angles; therefore the base BC is equal to the base EF (I. 4).

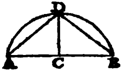


PROPOSITION XXX. PROBLEM.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc; it is required to bisect it.

Join AB, and bisect it in C (I. 10); from the point C draw CD at right angles to AB, and join AD, DB; the arc ADB is bisected in the point D.

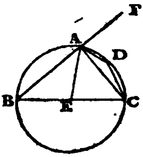


Because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides AC, CD, are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, because each of them is a right angle; therefore the base AD is equal to the base BD (I. 4). But equal chords cut off equal arcs (III. 28), the greater equal to the greater, and the less to the less; and AD, DB, are each of them less than a semicircle, because DC passes through the centre (III. 1, Cor.); wherefore the arc AD is equal to the arc DB; therefore the given arc is bisected in D.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA dividing the circle into the segments ABC, ADC, and join BA, AD, DC; the angle in the semicircle BAC is a right angle; and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.



Join AE, and produce BA to F; and because BE is equal to EA, the angle EAB is equal to EBA (I. 5); also, because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB. But FAC, the exterior angle of the triangle ABC, is also equal to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right angle (I. Def. 10); wherefore the angle BAC in a semicircle is a right angle.

And because the two angles ABC, BAC, of the triangle ABC are together less than two right angles (I. 17), and that BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC greater than a semicircle, is less than a right angle.

And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal to two right angles (III. 22); therefore the angles ABC, ADC, are equal to two right angles; and ABC is less than a right angle; wherefore the other ADC is greater than a right angle.

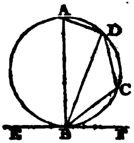
Cor.—From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two; and when the adjacent angles are equal, they are right angles.

PROPOSITION XXXII. THEOREM.

The angles contained by a tangent to a circle, and a chord drawn from the point of contact, are equal to the angles in the alternate segments of the circle.

Let EF be a tangent to the circle ABCD, and from the point of contact B let the chord BD be drawn cutting the circle; the angles which BD makes with the tangent EF shall be equal to the angles in the alternate segments of the circle; that is, the angle FBD is equal to the angle which is in the segment DAB, and the angle DBE to the angle in the segment BCD.

From the point B draw BA at right angles to EF (I. 11), and take any point C in the circumference BD, and join AD, DC, CB; and because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the tangent from the point of contact B, the centre of the circle is in BA (III. 19); therefore the angle ADB in a semicircle is a right angle (III. 31), and con-



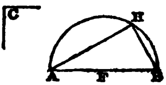
sequently the other two angles BAD, ABD, are equal to a right angle (I. 32); but ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD, ABD; take from these equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate segment of the circle; and because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD, are equal to two right angles (III. 22); therefore the angles DBF, DBE, being likewise equal to two right angles (I. 13), are equal to the angles BAD, BCD; and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle.

PROPOSITION XXXIII. PROBLEM.

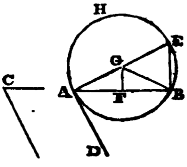
Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle; it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.

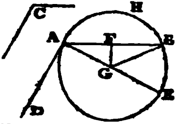
First, let the angle at C be a right angle, and bisect AB in F (I. 10), and from the centre F, at the distance FB, describe the semicircle AHB; therefore the angle AHB in a semicircle is equal to the right angle at C (III. 31).



But if the angle C be not a right angle, at the point A, in the straight line AB, make the angle BAD equal to the angle C (I. 23), and from the point A draw AE at right angles to AD; bisect AB in F (I. 11), and from F draw FG at right angles to AB, and join GB. And because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF, FG, are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal to the base GB (I. 4); and the circle described from the centre G, at the distance



GA, shall pass through the point B; let this be the circle AHB. And because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD touches the circle (III. 16, Cor.); and because AB drawn from the point of contact A cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (III. 32). But the angle DAB is equal to the angle C, therefore also the angle C is equal to the angle in the segment AHB. Wherefore, upon the given straight line AB the segment AHB of a circle is described which contains an angle equal to the given angle at C.

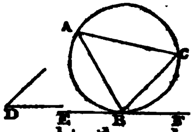


PROPOSITION XXXIV. PROBLEM.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a segment from the circle ABC that shall contain an angle equal to the angle D.

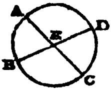
Draw the straight line EF touching the circle ABC in the point B (III. 17), and at the point B, in the straight line BF, make the angle FBC equal to the angle D (I. 23); therefore, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal to the angle in the alternate segment BAC of the circle (III. 32). But the angle FBC is equal to the angle D; therefore the angle in the segment BAC is equal to the angle D; wherefore the segment BAC is cut off from the given circle ABC containing an angle equal to the given angle D.



PROPOSITION XXXV. THEOREM.

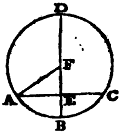
If two chords in a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two chords AC, BD, within the circle ABCD, cut one another in the point E; the rectangle contained by AE, EC, is equal to the rectangle contained by BE, ED.

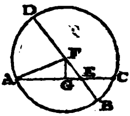


If AC, BD, pass each of them through the centre, so that E is the centre; it is evident that AE, EC, BE, ED, being all equal, the rectangle AE·EC, is likewise equal to the rectangle BE·ED.

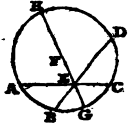
But let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, in the point E; then, if BD be bisected in F, F is the centre of the circle ABCD; join AF; and because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E; AE, EC, are equal to one another (III. 3); and because the straight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE·ED, together with the square of EF, is equal to the square of FB (II. 5); that is, to the square of FA; but the squares of AE, EF, are equal to the square of FA (I. 47); therefore the rectangle BE·ED, together with the square of EF, is equal to the squares of AE, EF. Take away the common square of EF, and the remaining rectangle BE·ED, is equal to the remaining square of AE; that is, to the rectangle AE·EC.



Next, let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles; then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw FG perpendicular to AC (I. 12); therefore AG is equal to GC (III. 3); wherefore the rectangle AE·EC, together with the square of EG, is equal to the square of AG. To each of these equals add the square of GF; therefore the rectangle AE·EC, together with the squares of EG, GF, is equal to the squares of AG, GF. But the squares of EG, GF, are equal to the square of EF; and the squares of AG, GF, are equal to the square of AF; therefore the rectangle AE·EC, together with the square of EF, is equal to the square of AF; that is, to the square of FB. But the square of FB is equal to the rectangle BE·ED, together with the square of EF; therefore the rectangle



AE·EC, together with the square of EF, is equal to the rectangle BE·ED, together with the square of EF. Take away the common square of EF, and the remaining rectangle AE·EC, is therefore equal to the remaining rectangle BE·ED.

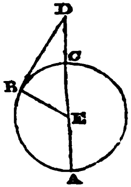


Lastly, let neither of the straight lines AC, BD, pass through the centre; take the centre F, and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH; and because the rectangle AE·EC, is equal, as has been shown, to the rectangle GE·EH; and, for the same reason, the rectangle BE·ED, is equal to the same rectangle GE·EH; therefore the rectangle AE·EC, is equal to the rectangle BE·ED.

PROPOSITION XXXVI. THEOREM.

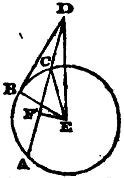
If from a point without a circle a secant and a tangent to it be drawn, the rectangle under the secant and its external segment is equal to the square of the tangent.

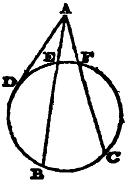
Let D be any point without the circle ABC, and DCA, DB, two straight lines drawn from it, of which DCA cuts the circle, and DB touches the same; the rectangle AD·DC, is equal to the square of DB.



Either DCA passes through the centre, or it does not; first, let it pass through the centre E, and join EB; therefore the angle EBD is a right angle (III. 18); and because the straight line AC is bisected in E, and produced to the point D, the rectangle AD·DC, together with the square of EC, is equal to the square of ED (II. 6); and CE is equal to EB; therefore the rectangle AD·DC, together with the square of EB, is equal to the square of ED; but (I. 47) the square of ED is equal to the squares of EB, BD, because EBD is a right angle; therefore the rectangle AD·DC, together with the square of EB, is equal to the squares of EB, BD; take away the common square of EB; therefore the remaining rectangle AD·DC, is equal to the square of the tangent DB.

But if DCA does not pass through the centre of the circle ABC, take the centre E (I. 3), and draw EF perpendicular to AC (I. 12), and join EB, EC, ED; and because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it shall likewise bisect it (III. 3); therefore AF is equal to FC; and because the straight line AC is bisected in F, and produced to D, the rectangle AD·DC, together with the square of FC, is equal to the square of FD; to each of these equals add the square of FE; therefore the rectangle AD·DC, together with the squares of CF, FE, is equal to the squares of DF, FE; but the square of ED is equal to the squares of DF, FE, because EFD is a right angle; and the square of EC is equal to the squares of CF, FE; therefore the rectangle AD·DC, together with the square of EC, is equal to the square of ED; and CE is equal to EB; therefore the rectangle AD·DC, together with the square of EB, is equal to the square of ED; but the squares of EB, BD, are equal to the square of ED, because EBD is a right angle; therefore the rectangle AD·DC, together with the square of EB, is equal to the squares of EB, BD; take away the common square of EB; therefore the remaining rectangle ADDC, is equal to the square of DB.





Cor.—If from any point without a circle there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another; namely, the rectangle BA·AE, to the rectangle CA·AF; for each of them is equal to the square of the straight line AD which touches the circle.

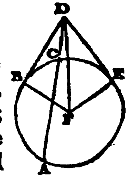
Schol. 1.—The above corollary and proposition 35, may be enunciated thus:—The rectangles under the segments of two intersecting chords of a circle are equal, whether they cut internally or externally.

Schol. 2.—The first case of this proposition affords a geometrical principle by which the length of the diameter of the earth may be computed.

PROPOSITION XXXVII. THEOREM.

If from a point without a circle a secant be drawn, and also a line meeting the circle, and if the rectangle under the secant and its external segment be equal to the square of the other line, this line will be a tangent.

Let any point D be taken without the circle ABC, and from it let two straight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD·DC, be equal to the square of DB, DB touches the circle.



Draw the straight line DE touching the circle ABC (III. 17); find the centre F, and join FE, FB, FD; then FED is a right angle (III. 18); and because DE touches the circle ABC, and DCA cuts it, the rectangle AD · DC, is equal to the square of DE (III. 36); but the rectangle AD · DC, is, by hypothesis, equal to the square of DB; therefore the square of DE is equal to the square of DB; and the straight line DE equal to the straight line DB; and FE is equal to FB, wherefore DE, EF, are equal to DB, BF; and the base FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal to the angle DBF (I. 8); and DEF is a right angle; therefore also DBF is a right angle; but FB, if produced, is a diameter, and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches the circle (III. 16); therefore DB touches the circle ABC.

EXERCISES.

1. A line that bisects two parallel chords in a circle, is also perpendicular to them.
2. Parallel chords in a circle intercept equal arcs.
3. The exterior angle of a quadrilateral figure inscribed in a circle, is equal to the interior and opposite.
4. If two circles cut each other, the line joining the points of intersection is bisected perpendicularly by the line joining their centres.

5. If a tangent to a circle be parallel to a chord, the point of contact bisects the intercepted arc.

6. Two concentric circles intercept between them, two equal portions of a line cutting them both.

7. If a chord to the greater of two concentric circles be a tangent to the less, it is bisected in the point of contact.

8. If a circle be described on the radius of another circle, any chord in the latter, drawn from the point in which the circles meet, is bisected by the former.

9. If two circles touch one another, either internally or externally, the chords of the two arcs intercepted by two lines drawn through the point of contact, are parallel.

10. If two circles cut one another, and, from one of the points of intersection, two diameters be drawn, their other extremities, and the other point of contact, will be in one straight line.

11. If two chords in a circle intersect each other perpendicularly, the sum of the squares of their four segments is equal to the square of the diameter.

12. Perpendiculars, from the extremities of a diameter of a circle, upon any chord, cut off equal segments.

13. In a circle, the sum of the squares of two lines drawn from the extremities of a chord, to any point in a diameter parallel to it, is equal to the sum of the squares of the segments of the diameter.

14. Given the vertical angle, the base, and the altitude of a triangle, to construct it.

15. Given the vertical angle, the base, and the sum of the sides of a triangle, to construct it.

16. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square of the line joining the points of concourse, is equal to the sum of the squares of the two tangents from these points.

17. If the points of contact of two tangents to a circle be joined, any secant drawn from their intersection is divided into three segments such, that the rectangle under the secant and its middle segment is equal to that under its extreme segments.

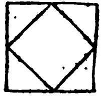
18. Two tangents to a circle, drawn from the same point, are equal.

FOURTH BOOK.

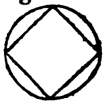
DEFINITIONS.

1. A rectilineal figure is said to be *inscribed in* another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

2. In like manner, a figure is said to be *described about* another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

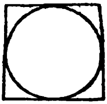


3. A rectilineal figure is said to be *inscribed in* a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

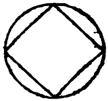


4. A rectilineal figure is said to be *described about* a circle when each side of the circumscribed figure touches the circumference of the circle.

5. In like manner, a circle is said to be *inscribed in* a rectilineal figure, when the circumference of the circle touches each side of the figure.



6. A circle is said to be *described about* a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



7. A straight line is said to be *placed in* a circle, when the extremities of it are in the circumference of the circle.

8. A *regular* polygon has all its sides equal, and also all its angles.

9. A polygon of five sides is called a *pentagon;* of six, a *hexagon;* of seven, a *heptagon;* of eight, an *octagon;* of nine, a *nonagon;* of ten, a *decagon;* of eleven, an *undecagon;* of twelve, a *dodecagon;* and of fifteen, a *quindecagon* or *pentedecagon.*

10. The *centre* of a regular polygon is a point equally distant from its sides or angular points.

11. The *apothem* of a regular polygon is a perpendicular from its centre upon any of its sides.

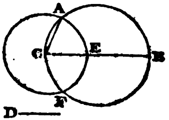
12. The *perimeter* of any figure is its circumference or whole boundary.

PROPOSITION I. PROBLEM.

In a given circle to place a straight line, equal to a given straight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle.

Draw BC a diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a straight line BC is placed equal to D; but if it is not, BC is greater than D; make CE equal to D (I. 3), and from the centre C, at the distance CE, describe the circle AEF, and join CA; therefore, because C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA. Wherefore, in the circle ABC, a straight line is placed, equal to the given straight line D, which is not greater than the diameter of the circle.

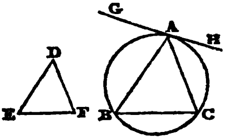


PROPOSITION II. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw the straight line GAH touching the circle in the point A (III. 17), and at the point A, in the straight line AH, make the angle HAC equal to the angle DEF (I. 23); and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC; therefore, because HAG touches the circle ABC, and AC is drawn from the point of contact, the angle

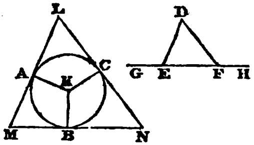


HAC is equal to the angle ABC in the alternate segment of the circle (III. 32); but HAC is equal to the angle DEF; therefore also the angle ABC is equal to DEF; for the same reason, the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal to the remaining angle EDF (I. 32); wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC.

PROPOSITION III. PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to describe a triangle about the circle ABC equiangular to the triangle DEF.



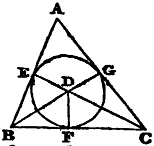
Produce EF both ways to the points G, H, and find the centre K of the circle ABC, and from it draw any straight line KB; at the point K in the straight line KB, make the angle BKA equal to the angle DEG (I. 23), and the angle BKC equal to the angle DFH; and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching the circle ABC (III. 17); therefore, because LM, MN, NL, touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC, the angles at the points A, B, C, are right angles (III. 18); and because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles, and because two of them KAM, KBM, are right angles, the other two AKB, AMB, are equal to two right angles. But the angles DEG, DEF, are likewise equal to two right angles (I. 13); therefore the angles AKB, AMB, are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF. In like manner, the angle LNM may be demonstrated to be equal to DFE; and therefore the remaining angle MLN is equal to the remaining angle EDF (I. 32); wherefore the triangle LMN is equiangular to the triangle DEF; and it is described about the circle ABC.

PROPOSITION IV. PROBLEM.

To inscribe a circle in a given triangle.

Let the given triangle be ABC; it is required to inscribe a circle in ABC.

Bisect the angles ABC, BCA, by the straight lines BD, CD, meeting one another in the point D, from which draw DE, DF, DG, perpendiculars to AB, BC, CA. And because the angle EBD is equal to the angle FBD, the angle ABC being bisected by BD; and because the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD, have two angles of the one equal to two angles of the other; and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore their other sides are equal (I.26); wherefore DE is equal to DF. For the same reason, DG is equal to DF; therefore the three straight lines DE, DF, DG, are equal to one another, and the circle described from the centre D, at the distance of any of them, will pass through the extremities of the other two, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G, are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it, touches the circle (III. 16); therefore the straight lines AB, BC, CA, do each of them touch the circle, and the circle EFG is inscribed in the triangle ABC.



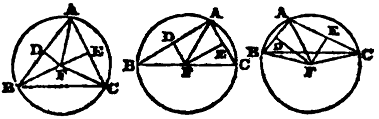
PROPOSITION V. PROBLEM.

To describe a circle about a given triangle.

Let the given triangle be ABC; it is required to describe a circle about ABC.

Bisect (I. 10) AB, AC, in the points D, E, and from these points draw DF, EF, at right angles to AB, AC (I. 11); DF, EF, produced, meet one another; for, if they do not meet, they are parallel; wherefore AB, AC, which are at right angles to them, are parallel, which is absurd.

Let them meet in F, and join FA; also, if the point F be not in BC, join BF, CF; then, because AD is equal to DB, and DF common, and at right angles to AB, the base AF is equal to the base FB (I. 4). In like manner, it may be shown that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC, are equal to one another; wherefore the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other two, and be described about the triangle ABC.



Cor.—And it is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle. Wherefore, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the hypotenuse; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.

Let ABCD be the given circle; it is required to inscribe a square in ABCD.

Draw the diameters AC, BD, at right angles to one another, and join AB, BC, CD, DA; because BE is equal to ED, E being the centre, and because EA is at right angles to BD, and common to the triangles ABE, ADE; the base BA is equal to the base AD (I. 4); and, for the same reason, BC, CD, are each of them equal to BA or AD; therefore the quadrilateral

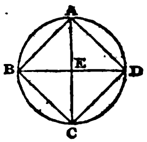
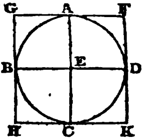


figure ABCD is equilateral. It is also rectangular; for the straight line BD, being the diameter of the circle ABCD, BAD is a semicircle; wherefore the angle BAD is a right angle (III. 31); for the same reason, each of the angles ABC, BCD, CDA, is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been shown to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD.

PROPOSITION VII. PROBLEM.

To describe a square about a given circle.

Let ABCD be the given circle; it is required to describe a square about it.



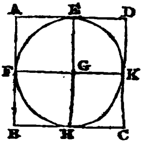
Draw two diameters AC, BD, of the circle ABCD, at right angles to one another, and (III. 17) through the points A, B, C, D, draw FG, GH, HK, KF, touching the circle; and because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right angles (III. 18); for the same reason, the angles at the points B, C, D, are right angles; and because the angle AEB is a right angle, as likewise is EBG, GH is parallel to AC (I. 28); for the same reason, AC is parallel to FK, and in like manner GF, HK, may each of them be demonstrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK, are parallelograms; and GF is therefore equal to HK (I. 34), and GH to FK; and because AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK; GH, FK, are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB is likewise a right angle. In the same manner, it may be shown that the angles at H, K, F, are right angles; therefore the quadrilateral figure FGHK is rectangular; and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD.

PROPOSITION VIII. PROBLEM.

To inscribe a circle in a given square.

Let ABCD be the given square; it is required to inscribe a circle in ABCD.

Bisect (I. 10) each of the sides AB, AD, in the points F, E, and through E draw EH parallel to AB or DC (I. 31), and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD, is a parallelogram, and their opposite sides are equal (I. 34); and because AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the sides opposite to these are equal, namely, FG to GE; in the same manner, it may be demonstrated, that GH, GK, are each of them equal to FG or GE; therefore the four straight lines GE, GF, GH, GK, are equal to one another; and the circle described from the centre G, at the distance of one of them, shall pass through the extremities of the other three; and shall also touch the straight lines AB, BC, CD, DA, because the angles at the points E, F, H, K, are right angles (I. 29), and because the straight line which is drawn from the extremity of a radius, at right angles to it, touches the circle (III. 16); therefore each of the straight lines AB, BC, CD, DA, touches the circle, which therefore is inscribed in the square ABCD.

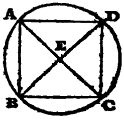


PROPOSITION IX. PROBLEM.

To describe a circle about a given square.

Let ABCD be the given square; it is required to describe a circle about it.

Join AC, BD, cutting one another in E; and because DA is equal to AB, and AC common to the triangles DAC, BAC, the two sides DA, AC, are equal to the two BA, AC, and the base DC is equal to the base BC; wherefore the angle DAC is equal to the angle BAC (I. 8), and the angle DAB is bisected by the straight line AC. In the same manner, it may be demonstrated that the angles ABC,

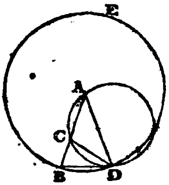


BCD, CDA, are severally bisected by the straight lines BD, AC; therefore, because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the side EA is equal to the side EB (I. 6). In the same manner, it may be demonstrated that the straight lines EC, ED, are each of them equal to EA or EB; therefore the four straight lines EA, EB, EC, ED, are equal to one another; and the circle described from the centre E, at the distance of one of them, shall pass through the extremities of the other three, and be described about the square ABCD.

PROPOSITION X. PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and (II. 11) divide it in the point C, so that the rectangle AB · BC may be equal to the square of CA; and from the centre A, at the distance AB, describe the circle BDE, in which place the straight line BD equal to AC (IV. 1), which is not greater than the diameter of the circle BDE; join DA, DC, and about the triangle ADC describe the circle ACD (IV. 5); the triangle ABD is such as is required; that is, each of the angles ABD, ADB, is double of the angle BAD.



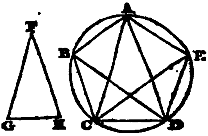
Because the rectangle AB · BC is equal to the square of AC, and that AC is equal to BD, the rectangle AB · BC is equal to the square of BD; and because from the point B without the circle ACD two straight lines BCA, BD, are drawn to the circumference, one of which cuts, and the other meets, the circle, and that the rectangle AB · BC contained by the secant, and the part of it without the circle, is equal to the square of BD which meets it; the straight line BD touches the circle ACD (III. 37); and because BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal to the angle DAC in the alternate segment of the circle (III. 32); to each of these add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA, DAC. But the exterior angle BCD is equal to the angles CDA, DAC (I. 32); therefore also BDA is equal to BCD; but (I. 5) BDA is equal to CBD, because the side AD is equal to the side AB; therefore CBD, or DBA, is equal to BCD; and consequently the three angles BDA, DBA, BCD, are equal to one another. And because the angle DBC is equal to the angle BCD, the side BD is equal to the side DC (I. 6). But BD was made equal to CA; therefore also CA is equal to CD, and the angle CDA equal to the angle DAC; therefore the angles CDA, DAC, together, are double of the angle DAC. But BCD is equal to the angles CDA, DAC; therefore also BCD is double of DAC; and BCD is equal to each of the angles BDA, DBA; each, therefore, of the angles BDA, DBA, is double of the angle DAB; wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle.

PROPOSITION XI. PROBLEM.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle, it is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.

Describe (IV. 10) an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F; and (IV. 2) in the circle ABCDE inscribe the triangle ACD equiangular to the triangle FGH, so that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA, equal to the angle at G or H; wherefore each of the angles ACD, CDA, is double of the angle CAD. Bisect the angles ACD, CDA (I. 9), by the straight lines CE, DB, and join AB, BC, DE, EA. ABCDE is the pentagon required.



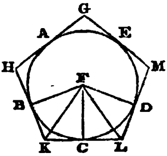
Because the angles ACD, CDA, are each of them double of CAD, and are bisected by the straight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA, are equal to one another, But (III. 26) equal angles stand upon equal arcs; therefore the five arcs AB, BC, CD, DE, EA, are equal to one another. And equal arcs are subtended by equal chords (III. 29); therefore the five chords AB, BC, CD, DE, EA, are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is also equiangular, because the arc AB is equal to the arc DE; if to each be added BCD, the whole ABCD is equal to the whole EDCB; and the angle AED stands on the arc ABCD, and the angle BAE on the arc EDCB; therefore the angle BAE is equal to the angle AED (III. 27). For the same reason, each of the angles ABC, BCD, CDE, is equal to the angle BAE, or AED; therefore the pentagon ABCDE is equiangular; and it has been shown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed

PROPOSITION XII. PROBLEM.

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to describe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, E, so that the arcs AB, BC, CD, DE, EA, are equal (IV. 11); and through the points A, B, C, D, E, draw GH, HK, KL, LM, MG, touching the circle (III. 17). Take the centre F, and join FB, FK, FC, FL, FD; and because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular to KL (III. 18); therefore each of the angles at C is a right angle. For the same reason, the angles at the points B, D, are right angles. And because FCK is a right angle, the square of FK is equal to the squares of FC, CK (I. 47). For the same reason, the square of FK is equal to the squares of FB, BK; therefore the squares of FC, CK, are



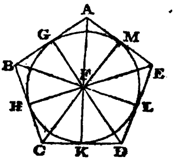
equal to the squares of FB, BK, of which the square of FC is equal to the square of FB; the remaining square of CK is therefore equal to the remaining square of BK, and the straight line CK equal to BK. And because FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK, are equal to the two CF, FK; and the base BK is equal to the base KC; therefore (I. 8) the angle BFK is equal to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC. For the same reason, the angle CFD is double of the angle CFL, and CLD double of CLF. And because the arc BC is equal to the arc CD, the angle BFC is equal to the angle CFD (III. 27); and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL; therefore in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each; and the side FC, which is adjacent to the equal angles *in each*, is common to both; therefore the other sides shall be equal to the other sides (I. 26), and the third angle to the third angle; therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC; and because KC is equal to CL, KL is double of KC. In the same manner, it may be shown that HK is double of BK; and because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK, HK shall be equal to KL. In like manner, it may be shown that GH, GM, ML, are each of them equal to HK or KL; therefore the pentagon GHKLM is equilateral. It is also equiangular; for, since the angle FKC is equal to the angle FLC, and the angle HKL double of the angle FKC, and KLM double of FLC, as was before demonstrated, the angle HKL is equal to KLM. And in like manner it may be shown, that each of the angles KHG, HGM, GML, is equal to the angle HKL or KLM; therefore the five angles GHK, HKL, KLM, LMG, MGH, being equal to one another, the pentagon GHKLM is equiangular; and it is equilateral, as was demonstrated; and it is described about the circle ABCDE.

PROPOSITION XIII. PROBLEM.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inscribe a circle in the pentagon ABCDE.

Bisect (I. 9) the angles BCD, CDE, by the straight lines CF, DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE; therefore, since BC is equal to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF, are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal to the base FD (I. 4), and the other angles to the other angles, to which the equal sides are opposite; therefore the angle CBF is equal to the angle CDF. And because the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is also double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is bisected by the straight line BF. In the same manner, it may be demonstrated that the angles BAE, AED, are bisected by the straight lines AF, FE. From the point F draw (I. 12) FG, FH, FK, FL, FM, perpendiculars to the straight lines AB, BC, CD, DE, EA; and because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC, FKC, there are two angles of one equal to two angles of the other, and the side FC, which is opposite to one of the equal angles in each, is common to both; therefore the other sides shall be equal, each to each (I. 26); wherefore the perpendicular FH is equal to the perpendicular FK. In the same manner, it may be demonstrated that FL, FM, FG, are each of them equal to FH or FK; therefore the five straight lines FG, FH, FK, FL, FM, are equal to one another; wherefore the circle described from the centre F, with any of these as a radius, shall pass through the extremities of the other four, and touch the straight



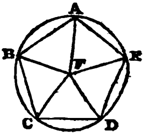
lines AB, BC, CD, DE, EA, because that the angles at the points G, H, K, L, M, are right angles, and that a straight line drawn from the extremity of the diameter of a circle at right angles to it, touches the circle; therefore each of the straight lines AB, BC, CD, DE, EA, touches the circle; wherefore it is inscribed in the pentagon ABCDE.

PROPOSITION XIV. PROBLEM.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to describe a circle about it.

Bisect (I. 9) the angles BCD, CDE, by the straight lines CF, FD, and from the point F, in which they meet, draw the straight lines FB, FA, FE, to the points B, A, E. It may be demonstrated, in the same manner as in the preceding proposition, that the angles CBA, BAE, AED, are bisected by the straight lines FB, FA, FE; and because the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the side CF is equal to the side FD (I. 6). In like manner, it may be demonstrated that FB, FA, FE, are each of them equal to FC or FD; therefore the five straight lines FA, FB, FC, FD, FE, are equal to one another; and the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon ABCDE.



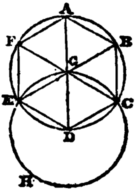
PROPOSITION XV. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inscribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the distance

DG, describe the circle EGCH; join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA; the hexagon ABCDEF is equilateral and equiangular.



Because G is the centre of the circle ABCDEF, GE is equal to GD; and because D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG, are equal to one another (I. 5, Cor.); and the three angles of a triangle are equal to two right angles (I. 32); therefore the angle EGD is the third part of two right angles. In the same manner, it may be demonstrated that the angle DGC is also the third part of two right angles; and because the straight line GC makes with EB the adjacent angles EGC, CGB, equal to two right angles (I. 13); the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB, are equal to one another; and to these are equal the vertical opposite angles BGA, AGF, FGE (III. 15); therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. But equal angles stand upon equal circumferences (III. 26); therefore the six arcs AB, BC, CD, DE, EF, FA, are equal to one another; and equal arcs are subtended by equal chords (III. 29); therefore the six chords are equal to one another, and the hexagon ABCDEF is equilateral. It is also equiangular; for, since the arc AF is equal to ED, to each of these add the arc ABCD; therefore the whole arc FABCD shall be equal to the whole EDCBA; and the angle FED stands upon the arc FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED. In the same manner, it may be demonstrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED; therefore the hexagon is equiangular; and it is equilateral, as was shown; and it is inscribed in the given circle ABCDEF.

Cor.—From this it is manifest, that the side of the hexagon is equal to the radius of the circle.

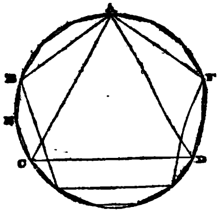
And if through the points A, B, C, D, E, F, there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

PROPOSITION XVI. PROBLEM.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the side of an equilateral triangle inscribed in the circle (IV. 2), and AB the side of an equilateral and equiangular pentagon inscribed in the same (IV. 11); therefore, of such equal parts as the whole circumference ABCDF contains fifteen, the arc ABC, being the third part of the whole, contains five; and the arc AB, which is the fifth part of the whole, contains three; therefore BC their difference contains two of the same parts; bisect BC in E (III. 30); therefore BE, EC, are, each of them, the fifteenth part of the whole circumference ABCD: therefore, if the straight lines BE, EC, be drawn, and (IV. 1) straight lines equal to them be placed around in the whole circle, an equilateral and equiangular quindecagon shall be inscribed in it.



And in the same manner, as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it. And likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

Schol.—Whatever regular polygon can be inscribed in a circle, another of double the number of sides may be inscribed in it, by bisecting the arc subtended by a side of the former. It has been proved by M. Gauss, that it is practicable by means of plane geometry, that is, by the intersections of the straight line and circle, or by the resolution of simple and quadratic equations, to inscribe in a circle any regular polygon, provided the number of its sides be a prime* number, and be also some power of 2 increased by 1.

EXERCISES.

1. To describe a circle that shall touch a given line in a given point, and pass through another given point.
2. To draw a line that shall be a common tangent to two circles.
3. To describe a circle that shall pass through two given points, and touch a given line.
4. To describe a circle that shall touch a given line in a given point, and also touch a given circle.
5. To draw a tangent to a given circle, so that it shall be parallel to a given line.
6. To draw a tangent to a circle, so that it shall make a given angle with a given line.
7. A regular octagon inscribed in a circle is equal to the rectangle under the sides of the inscribed and circumscribing squares.
8. To inscribe a circle in a given quadrant.
9. To describe a circle that shall pass through a given point, and touch a given circle in a given point.
10. If a quadrilateral circumscribe a circle, its opposite sides are together equal to half its perimeter.

FIFTH BOOK.

DEFINITIONS.

1. A less magnitude is said to be a *part* of a greater magnitude, when the less *measures* the greater; that is, when the less is contained a certain number of times exactly in the greater. It is also called a *measure*, or *submultiple* of the greater.

---

* That is, a number which has no common measure except 1.

2. A line, which is a measure of two or more lines, is called a *common measure* of these lines.

3. Lines that have a common measure, are called *commensurable* lines; those that have no common measure, *incommensurable* lines.

4. A greater magnitude is said to be a *multiple* of a less, when the greater is *measured* by the less; that is, when the greater contains the less a certain number of times exactly.

5. *Equimultiples* of magnitudes are multiples that contain them, respectively, the same number of times.

6. *Ratio* is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.

7. Magnitudes are said to be *homogeneous*, or of the *same kind*, when the less can be multiplied so as to exceed the greater; and it is only such magnitudes that are said to have a ratio to one another.

8. The two magnitudes of a ratio are called its *terms*. The first term is called the *antecedent;* the latter, the *consequent*.

9. A ratio is said to be a ratio of *equality*, *majority*, or *minority*, according as the antecedent is equal to, greater or less than, the consequent.

10. If there be four magnitudes, and if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, and if, according as the multiple of the *first* is greater than the multiple of the *second*, equal to it, or less, the multiple of the *third* is also greater than the multiple of the *fourth*, equal to it, or less; then—the *first* of the magnitudes is said to have to the *second* the *same ratio* that the *third* has to the *fourth*.

11. Magnitudes are said to be *proportionals*, when the first has the same ratio to the second that the third has to the fourth; and the third to the fourth the same ratio which the fifth has to the sixth, and so on, whatever be their number.

When four magnitudes, A, B, C, D, are proportionals, it is usual to say that A is to B as C to D, and to write them thus $A : B :: C : D$, or thus, $A : B = C : D$.

12. When four magnitudes are proportional, they constitute a *proportion* or an *analogy*.

13. The first and last terms of an analogy are called the *extremes;* the second and third, the *means*.

14. When of the equimultiples of four magnitudes, taken as in the tenth definition, the multiple of the first is *greater* than that of the second, but the multiple of the third is *not greater* than the multiple of the fourth; then the first is said to have to the second a *greater ratio* than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a *less ratio* than the first has to the second.

15. When there is any number of magnitudes greater than two, of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio which the third has to the fourth, and so on, the magnitudes are said to be *continual proportionals*, or *in continued proportion*.

16. When three magnitudes are continual proportionals, the second is said to be a *mean proportional* between the other two.

17. When there is any number of magnitudes of the same kind, the *first* is said to have to the *last* of them the *ratio compounded* of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude.

For example, if A, B, C, D, be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A : B :: E : F; and B : C :: G : H, and C : D :: K : L, then, since by this definition A has to D the ratio compounded of the ratios of A to B, B to C, C to D; A may also be said to have to D the *ratio compounded* of the ratios which are the *same* with the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D, then, for shortness sake, M is said to have to N a ratio *compounded of the same ratios* which compound the ratio of A to D; that is, a ratio compounded of the ratios of E to F, G to H, and K to L.

18. A ratio which is compounded of *two equal ratios*, is said to be *duplicate* of either of these ratios.

Cor.—Hence, if the three magnitudes A, B, and C, are continual proportionals, the ratio of A to C is duplicate of that of A to B, or of B to C; for, by the last definition, the ratio of A to C is compounded of the ratios of A to B, and of B to C; but the ratio of A to B is equal to the ratio of B to C, because A, B, C, are continual proportionals; therefore the ratio of A to C, by this definition, is duplicate of the ratio of A to B, or of B to C.

19. A ratio which is compounded of *three equal ratios* is said to be *triplicate* of any one of these ratios; and a ratio which is compounded of *four equal ratios*, is said to be *quadruplicate* of any one of these ratios; and so on, according to the number of equal ratios.

Cor.—If four magnitudes, A, B, C, D, be continual proportionals, the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D. For the ratio of A to D is compounded of the three ratios of A to B, B to C, C to D; and these three ratios are equal to one another, because A, B, C, D, are continual proportionals; therefore the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D.

20. In proportionals, the antecedent terms are called *homologous* to one another, as also the consequents to one another.

Geometers make use of the following technical words to signify certain ways of changing either the order or magnitude of the terms of an analogy, so that they continue still to be proportionals:—

21. By *alternation*, when the first is to the third, as the second is to the fourth.

22. By *inversion*, when the second is to the first, as the fourth is to the third.

23. By *composition*, when the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth.

24. By *addition*, when the first is to the sum of the first and second, as the third is to the sum of the third and fourth.

25. By *division*, when the excess of the first above the

second, is to the second, as the excess of the third above the fourth, is to the fourth.

26. By *conversion*, when the first is to its excess above the second, as the third is to its excess above the fourth.

27. By *mixing*, when the sum of the first and second, is to their difference, as the sum of the third and fourth to their difference.

28. By *equality*, when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred—that the *first* is to the *last* of the *first rank* of magnitudes, as the *first* is to the *last* of the *others*:—of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two:

29. By *direct equality*, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in a *direct order*: and,

30. By *indirect equality*, when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in an *indirect order*.

The following are some additional algebraical expressions which are convenient for their conciseness.

As mA is A taken m times, it is a multiple of A by m. So m (A + B) is a multiple of A + B by m; m (A — B), a multiple of A — B by m; and m (A + B — C), a multiple of the excess of A + B above C, by m.

Also, mA and mB are equimultiples of A and B by m.

The expression $m + n$ is the sum of the numbers m and n; so mn is the product of these numbers. Also, mnA is a multiple of A by a number equal to mn; and $(m + n)$ A, is a multiple of A by a number equal to $m + n$.

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

2. Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROPOSITION I. THEOREM.

If any number of magnitudes be equimultiples of as many others, each of each, what multiple soever any one of the first is of its part, the same multiple is the sum of all the first of the sum of all the rest.

Let any number of magnitudes A, B, and C, be *equimultiples* of as many others D, E, and F, each of each; $A + B + C$ is the same multiple of $D + E + F$, that A is of D.

Let A contain D, B contain E, and C contain F, each any number of times, as, for instance, three times. Then, because A contains D three times $A = D + D + D$.

For the same reason, $B = E + E + E$;

And also, $C = F + F + F$.

Therefore, adding equals to equals, $A + B + C$ is equal to $D + E + F$ taken three times. In the same manner, if A, B, and C, were each any other equimultiple of D, E, and F, it would be shown that $A + B + C$ was the same multiple of $D + E + F$.

Cor.—Hence, if m be any number, $mD + mE + mF = m(D + E + F)$. For mD, mE, and mF, are multiples of D, E, and F, by m; therefore their sum is also a multiple of $D + E + F$, by m.

PROPOSITION II. THEOREM.

If to a multiple of a magnitude by any number a multiple of the same magnitude by any number be added, the sum

will be the same multiple of that magnitude that the sum of the two numbers is of unity.

Let $A = mC$, and $B = nC$; $A + B = (m + n) C$.

For, since $A = mC$, $A = C + C + C +$ &c. C being repeated m times. For the same reason, $B = C + C +$ &c, C being repeated n times. Therefore, adding equals to equals, $A + B$ is equal to C taken $m + n$ times; that is, $A + B = (m + n) C$. Therefore $A + B$ contains C as oft as there are units in $m + n$.

COR. 1.—In the same way, if there be any number of multiples whatsoever, as $A = mE$, $B = nE$, $C = pE$, it is shown that $A + B + C = (m + n + p) E$.

COR. 2.—Hence also, since $A + B + C = (m + n + p) E$, and since $A = mE$, $B = nE$, and $C = pE$, $mE + nE + pE = (m + n + p) E$.

PROPOSITION III. THEOREM.

If the first of three magnitudes contain the second as oft as there are units in a certain number, and if the second contain the third also as often as there are units in a certain number, the first will contain the third as oft as there are units in the product of these two numbers.

Let $A = mB$, and $B = nC$; then $A = mnC$.

Since $B = nC$, $mB = nC + nC +$ &c. repeated m times. But $nC + nC +$ &c. repeated m times, is (V. 2) equal to C multiplied by $n + n +$ &c. n being repeated m times. And n repeated m times is mn; therefore $mB = mnC$. But by hypothesis, $A = mB$, therefore $A = mnC$.

PROPOSITION IV. THEOREM.

If any equimultiples be taken of the antecedents of an analogy, and any equimultiples of the consequents, these multiples, taken in the order of the terms, are proportional.

Let $A : B :: C : D$, and let m and n be any two numbers; $mA : nB :: mC : nD$.

Take of mA and mC equimultiples by any number p, and of nB and nD equimultiples by any number q. Then the equimultiples of mA and mC by p, are equimultiples also of A and C, for they contain A and C as oft as there are units in pm (V. 3), and are equal to pmA and pmC. For the same reason, the multiples of nB and nD by q, are qnB, qnD. Since therefore A : B : : C : D, and of A and C there are taken any equimultiples; namely, pmA and pmC, and of B and D, any equimultiples qnB, qnD, if pmA be greater than qnB, pmC must be greater than qnD (V. Def. 10); if equal, equal; and if less, less. But pmA, pmC, are also equimultiples of mA and mC, and qnB, qnD, are equimultiples of nB and nD, therefore mA : nB : : mC : nD.

COR.—In the same manner, it may be demonstrated that if A : B : : C : D, and of A and C equimultiples be taken by any number m; namely, mA and mC, mA : B : : mC : D. This may also be considered as included in the proposition, and as being the case when $n = 1$.

PROPOSITION V. THEOREM.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let mA and mB be any equimultiples of the two magnitudes A and B, of which A is greater than B; $m\text{A} - m\text{B}$ is the same multiple of $\text{A} - \text{B}$ that mA is of A, that is, $m\text{A} - m\text{B} = m\,(\text{A} - \text{B})$.

Let D be the excess of A above B, then $\text{A} - \text{B} = \text{D}$, and adding B to both $\text{A} = \text{D} + \text{B}$. Therefore (V. 1) $m\text{A} = m\text{D} + m\text{B}$; take mB from both, and $m\text{A} - m\text{B} = m\text{D}$; but $\text{D} = \text{A} - \text{B}$, therefore $m\text{A} - m\text{B} = m\,(\text{A} - \text{B}.)$

PROPOSITION VI. THEOREM.

If from a multiple of a magnitude by any number a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude that the difference of the numbers is of unity.

Let mA and nA be multiples of the magnitude A by the numbers m and n, and let m be greater than n; $m\text{A} - n\text{A}$ contains A as oft as $m - n$ contains unity, or $m\text{A} - n\text{A} = (m - n)\,\text{A}$.

Let $m - n = q$; then $m = n + q$. Therefore (V. 2)

$mA = nA + qA$; take nA from both, and $mA - nA = qA$. Therefore $mA - nA$ contains A as oft as there are units in q, that is, in $m - n$, or $mA - nA = (m - n)\ A$.

Cor.—When the difference of the two numbers is equal to unity, or $m - n = 1$, then $mA - nA = A$.

PROPOSITION A. THEOREM.

If four magnitudes be proportionals, they are proportionals also when taken inversely.

If A : B : : C : D, then also B : A : : D : C.

Let mA and mC be any equimultiples of A and C; nB and nD any equimultiples of B and D. Then, because A : B : : C : D, if mA be less than nB, mC will be less than nD (V. Def. 10), that is, if nB be greater than mA, nD will be greater than mC. For the same reason, if $nB = mA$, $nD = mC$, and if $nB \angle mA$, $nD \angle mC$. But nB, nD, are any equimultiples of B and D, and mA, mC, any equimultiples of A and C, therefore B : A : : D : C.

PROPOSITION B. THEOREM.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second as the third to the fourth.

First, if mA, mB, be equimultiples of the magnitudes A and B, mA : A : : mB : B.

Take of mA and mB equimultiples by any number n; and of A and B equimultiples by any number p; these will be nmA (V. 3), pA, nmB, pB. Now, if nmA be greater than pA, nm is also greater than p; and if nm is greater than p, nmB is greater than pB; therefore, when nmA is greater than pA, nmB is greater than pB. In the same manner, if $nmA = pA$, $nmB = pB$, and if $nmA \angle pA$, $nmB \angle pB$. Now, nmA, nmB, are any equimultiples of mA and mB; and pA, pB, are any equimultiples of A and B, therefore mA : A : : mB : B (V. Def. 10).

Next, let C be the same part of A that D is of B; then A is the same multiple of C that B is of D; and therefore, as has been demonstrated, A : C : : B : D, and inversely (V. A) C : A : : D : B.

PROPOSITION C. THEOREM.

If the first term of an analogy be a multiple or a part of the second, the third is the same multiple or the same part of the fourth.

Let A : B : : C : D, and first let A be a multiple of B, C is the same multiple of D; that is, if $A = mB$, $C = mD$.

Take of A and C equimultiples by any number as 2; namely, 2A and 2C; and of B and D, take equimultiples by the number $2m$, namely, $2mB$, $2mD$ (V. 3); then, because $A = mB$, $2A = 2mB$; and since A : B : : C : D, and since $2A = 2mB$, therefore $2C = 2mD$ (V. Def. 10), and $C = mD$, that is, C contains D m times, or as often as A contains B.

Next, let A be a part of B, C is the same part of D. For, since A : B : : C : D, inversely (V. A), B : A : : D : C. But A being a part of B, B is a multiple of A, and therefore, as is shown above, D is the same multiple of C, and therefore C is the same part of D that A is of B.

PROPOSITION VII. THEOREM.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other; A : C : : B : C.

Let mA, mB, be any equimultiples of A and B; and nC any multiple of C.

Because $A = B$, $mA = mB$ (V. Ax. 1); wherefore, if mA be greater than nC, mB is greater than nC (V. Def. 10); and if $mA = nC$, $mB = nC$; or, if $mA \angle nC$, $mB \angle nC$. But mA and mB are any equimultiples of A and B, and nC is any multiple of C, therefore A : C : : B : C.

Again, if $A = B$, C : A : : C : B; for, as has been proved, A : C : : B : C, and inversely (V. A), C : A : : C : B.

PROPOSITION VIII. PROBLEM.

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Let $A + B$ be a magnitude greater than A and C a third magnitude, $A + B$ has to C a greater ratio than A has to C; and C has a greater ratio to A than it has to $A + B$.

Let m be such a number that mA and mB are each of them greater than C; and let nC be the least multiple of C that exceeds $mA + mB$; then $nC - C$, that is (V. 1), $(n - 1) C$ will be less than $mA + mB$, or $mA + mB$, that is, $m (A + B)$ is greater than $(n - 1) C$. But because nC is greater than $mA + mB$, and C less than mB, $nC - C$ is greater than mA, or mA is less than $nC - C$, that is, than $(n - 1) C$. Therefore the multiple of $A + B$ by m exceeds the multiple of C by $n - 1$, but the multiple of A by m does not exceed the multiple of C by $n - 1$; therefore $A + B$ has a greater ratio to C than A has to C (V. Def. 14).

Again, because the multiple of C by $n - 1$, exceeds the multiple of A by m, but does not exceed the multiple of $A + B$ by m, C has a greater ratio to A than it has to $A + B$.

PROPOSITION IX. THEOREM.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

If $A : C :: B : C$, $A = B$.

For, if not, let A be greater than B; then, because A is greater than B, two numbers, m and n, may be found as in the last proposition, such that mA shall exceed nC, while mB does not exceed nC. But because $A : C :: B : C$; if mA exceed nC, mB must also exceed nC (V. Def. 10); and it is also shown that mB does not exceed nC, which is impossible. Therefore A is not greater than B; and in the same way it is demonstrated that B is not greater than A; therefore A is equal to B.

Next, let $C : A :: C : B$, $A = B$. For by inversion (V. A), $A : C :: B : C$; and therefore by the first case, $A = B$.

PROPOSITION X. THEOREM.

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and

that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

If the ratio of A to C be greater than that of B to C, A is greater than B.

Because $A : C > B : C$, two numbers m and n may be found, such that $mA > nC$, and $mB < nC$ (V. Def. 14). Therefore also $mA > mB$, and $A > B$ (IV. Ax. 5).

Again, let $C : B > C : A$; $B < A$. For two numbers, m and n may be found, such that $mC > mB$, and $mC < mA$. Therefore, since mB is less, and mA greater than the same magnitude, mC, $mB < mA$, and therefore $B < A$.

PROPOSITION XI. THEOREM.

Ratios that are equal to the same ratio are equal to one another.

If $A : B :: C : D$; and also $C : D :: E : F$; then $A : B :: E : F$.

Take mA, mC, mE, any equimultiples of A, C, and E; and nB, nD, nF, any equimultiples of B, D, and F. Because $A : B :: C : D$, if $mA > nB$, $mC > nD$ (V. Def. 10); but if $mC > nD$, $mE > nF$, because $C : D :: E : F$; therefore if $mA > nB$, $mE > nF$. In the same manner, if $mA = nB$, $mE = nF$; and if $mA < nB$, $mE < nF$. Now, mA, mE, are any equimultiples whatever of A and E; and nB, nF, any whatever of B and F; therefore $A : B :: E : F$ (V. Def. 10).

PROPOSITION XII. THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so is the sum of all the antecedents to that of the consequents.

If $A : B :: C : D$, and $C : D :: E : F$; $A : B :: A + C + E : B + D + F$.

Take mA, mC, mE, any equimultiples of A, C, and E; and nB, nD, nF, any equimultiples of B, D, and F. Then, because $A : B :: C : D$, if $mA > nB$, $mC > nD$ (V. Def. 8); and when $mC > nD$, $mE > nF$, because $C : D :: E : F$. Therefore, if $mA > nB$, $mA + mC + mE > nB + nD + nF$. In the same manner, if $mA = nB$, $mA + mC + mE =$

$nB + nD + nF$; and if $mA < nB$, $mA + mC + mE < nB + nD + nF$. Now, $mA + mC + mE = m(A + C + E)$ (V. 1), so that mA and $mA + mC + mE$ are any equal multiples of A and of $A + C + E$. And for the same reason nB, and $nB + nD + nF$, are any equimultiples of B, and of $B + D + F$; therefore $A : B :: A + C + E : B + D + F$.

PROPOSITION XIII. PROBLEM.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

If $A : B :: C : D$; but $C : D > E : F$; then also, $A : B > E : F$.

Because $C : D > E : F$, there are some two numbers m and n, such that $mC > nD$, but $mE < nF$ (V. Def. 14). Now, if $mC > nD$, $mA > nB$, because $A : B :: C : D$. Therefore $mA > nB$, and $mE < nF$; wherefore $A : B > E : F$.

PROPOSITION XIV. THEOREM.

If the first term of an analogy be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

If $A : B :: C : D$; then if $A > C$, $B > D$, if $A = C$, $B = D$, and if $A < C$, $B < D$.

First, let $A > C$; then $A : B > C : B$ (V. 8), but $A : B :: C : D$, therefore $C : D > C : B$ (V. 13), and therefore $B > D$ (V. 10).

In the same manner, it is proved, that if $A = C$, $B = D$; and if $A < C$, $B < D$.

PROPOSITION XV. THEOREM.

Magnitudes have the same ratio to one another which their equimultiples have.

If A and B be two magnitudes, and m any number, $A : B :: mA : mB$.

Because $A : B :: A : B$ (V. 7); $A : B :: A + A : B + B$

(V. 12), or $A : B :: 2A : 2B$. And since it has been proved that $A : B :: 2A : 2B$, $A : B :: A + 2A :: B + 2B$ (V. 12), or $A : B :: 3A : 3B$; and so on, for all the equimultiples of A and B.

PROPOSITION XVI. THEOREM.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

If $A : B :: C : D$, then alternately, $A : C :: B : D$.

Take mA, mB, any equimultiples of A and B, and nC, nD, any equimultiples of C and D. Then (V. 15) $A : B :: mA : mB$; now $A : B :: C : D$; therefore (V. 11) $C : D :: mA : mB$. But $C : D :: nC : nD$ (V. 15); therefore $mA : mB :: nC : nD$ (V. 11); wherefore if $mA > nC$, $mB > nD$ (V. 14); if $mA = nC$, $mB = nD$, or if $mA < nC$, $mB < nD$; therefore, $A : C :: B : D$.

PROPOSITION XVII. THEOREM.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if the first, together with the second, have to the second the same ratio which the third, together with the fourth, has to the fourth, the first shall have to the second the same ratio which the third has to the fourth.

If $A + B : B :: C + D : D$, then by division $A : B :: C : D$.

Take mA and nB any equimultiples of A and B, by the numbers m and n; and first, let $mA > nB$; to each of them add mB, then $mA + mB > mB + nB$. But $mA + mB = m(A + B)$ (V. 1), and $mB + nB = (m + n)B$ (V. 2), therefore $m(A + B) > (m + n)B$.

And because $A + B : B :: C + D : D$, if $m(A + B) > m + n.B$, $m(C + D) > (m + n)D$, or $mC + mD > mD + nD$, that is, taking mD from both, $mC > nD$. Therefore, when mA is greater than nB, mC is greater than nD. In like manner it is demonstrated, that if $mA = nB$, $mC = nD$; and if $mA < nB$, $mC < nD$; therefore $A : B :: C : D$ (V. Def. 10).

PROPOSITION XVIII. THEOREM.

The terms of an analogy are proportional by composition.

If $A : B :: C : D$, then, by composition, $A + B : B :: C + D : D$.

Take $m\,(A+B)$ and nB any multiples whatever of $A+B$ and B: and first, let m be greater than n. Then, because $A + B$ is also greater than B, $m\,(A + B) > n.B$. For the same reason, $m\,(C + D) > nD$. In this case, therefore, that is, when $m > n$, $m\,(A + B)$ is greater than nB, and $m\,(C+D)$ is greater than nD. And in the same manner, it may be proved that when $m = n$, $m\,(A + B)$ is greater than nB, and $m\,(C + D)$ greater than nD.

Next, let $m < n$, or $n > m$, then $m\,(A + B)$ may be greater than nB, or may be equal to it, or may be less; first let $m\,(A + B)$ be greater than nB; then also, $mA + mB > nB$; take mB, which is less than nB, from both, and $mA > nB - mB$, or $mA > (n - m)\,B$ (V. 6). But if $mA > (n - m)\,B$, $mC > (n - m)\,D$; because $A : B :: C : D$. Now, $(n - m)\,D = nD - mD$, therefore $mC > nD - mD$, and adding mD to both $mC + mD > nD$, that is (V. 1) $m\,(C + D) > nD$. If therefore $m\,(A + B) > nB$, $m\,(C + D) > nD$.

In the same manner, it will be proved that if $m\,(A+B) = nB$, $m\,(C + D) = nD$; and if $m\,(A + B) < nB$, $m\,(C + D) < nD$; therefore $A + B : B :: C + D : D$.

PROPOSITION XIX. THEOREM.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

If $A : B :: C : D$, and if C be less than A, $A - C : B - D :: A : B$.

Because $A : B :: C : D$, alternately (V. 16), $A : C :: B : D$; and therefore by division (V. 17) $A - C : C :: B - D : D$. Wherefore, again, alternately, $A - C : B - D :: C : D$; but $A : B :: C : D$, therefore $A - C : B - D :: A : B$.

COR.—$A - C : B - D :: C : D$.

PROPOSITION D. THEOREM.

The terms of an analogy are proportional by conversion.

If $A : B :: C : D$, by conversion, $A : A - B :: C : C - D$.

For since $A : B :: C : D$, by division (V. 17), $A - B : B :: C - D : D$, and inversely (V. A), $B : A - B :: D : C - D$; therefore, by composition (V. 18), $A : A - B :: C : C - D$.

PROPOSITION XX. THEOREM.

If there be three magnitudes, and other three, which, taken two and two in order, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

If there be three magnitudes, A, B, and C, and other three D, E, and F; and if $A : B :: D : E$; and also $B : C :: E : F$, then if $A > C$, $D > F$; if $A = C$, $D = F$; and if $A < C$, $D < F$.

A,	B,	C,
D,	E,	F.

First, let $A > C$; then $A : B > C : B$ (V. 8). But $A : B :: D : E$, therefore also $D : E > C : B$ (V. 13). Now $B : C :: E : F$, and inversely (V. A), $C : B :: F : E$; and it has been shown that $D : E > C : B$, therefore $D : E > F : E$; and consequently $D > F$ (V. 10).

Next, let $A = C$; then $A : B :: C : B$ (V. 7) but $A : B :: D : E$, therefore, $C : B :: D : E$; but $C : B :: F : E$, therefore, $D : E :: F : E$ (V. 11), and $D = F$ (V. 9). Lastly, let $A < C$. Then $C > A$, and, as was shown in the first case, $C : B :: F : E$, and $B : A :: E : D$; therefore, by the first case, if $C > A$, $F > D$, that is, if $A < C$, $D < F$.

PROPOSITION XXI. THEOREM.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

If there be three magnitudes, A, B, C, and other three, D, E, and F, such that $A : B :: E : F$, and $B : C :: D : E$; if $A > C$, $D > F$; if $A = C$, $D = F$, and if $A < C$, $D < F$.

First, let $A > C$. Then $A : B > C : B$ (V. 8), but $A : B :: E : F$; therefore, $E : F > C : B$ (V. 13). Now, $B : C :: D : E$, and inversely, $C : B :: E : D$; therefore, $E : F > E : D$; wherefore, $D > F$ (V. 10).

A, B, C,
D, E, F.

Next, let $A = C$. Then (V. 7) $A : B :: C : B$; but $A : B :: E : F$; therefore, $C : B :: E : F$ (V. 11); but $B : C :: D : E$, and inversely, $C : B :: E : D$; therefore, $E : F :: E : D$, and consequently, $D = F$ (V. 9).

Lastly, let $A < C$. Then $C > A$, and, as was proved in the first case, $C : B :: E : D$; and $B : A :: F : E$; therefore, by the first case, since $C > A$, $F > D$, that is, $D < F$.

PROPOSITION XXII. THEOREM.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be three magnitudes, A, B, C, and other three, D, E, F, which, taken two and two in order, have the same ratio, namely, $A : B :: D : E$, and $B : C :: E : F$; then $A : C :: D : F$.

Take of A and D any equimultiples whatever, mA, mD; of B and E any whatever, nB, nE; and of C and F any whatever, qC, qF. Because $A : B :: D : E$, $mA : nB :: mD : nE$ (V. 4); and for the same reason, $nB : qC :: nE : qF$. Therefore (V. 20), according as mA is greater than qC, equal to it, or less, mB is greater than qF, equal to it, or less; but mA, mD, are any equimultiples of A and D; and qC, qF, are any equimultiples of C and F; therefore (V. Def. 10) $A : C :: D : F$.

A, B, C,
D, E, F,
mA, nB, qC,
mD, nE, qF.

Again, let there be four magnitudes, and other four, which, two and two, have the same ratio; namely, $A : B :: E : F$; $B : C :: F : G$; $C : D :: G : H$, then $A : D :: E : H$.

For since A, B, C, are three magnitudes and E, F, G, other three, which, taken two and two, have the same ratio, by the foregoing case $A : C :: E : G$. And because

A, B, C, D,
E, F, G, H.

also C : D : : G : H, by that same case, A : D : : E : H. In the same manner is the demonstration extended to any number of magnitudes.

PROPOSITION XXIII. THEOREM.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be three magnitudes, A, B, C, and other three, D, E, and F, which, taken two and two in a cross order, have the same ratio; namely, A : B : : E : F, and B : C : : D : E, then A : C : : D : F.

Take of A, B, and D, any equimultiples mA, mB, mD; and of C, E, F, any equimultiples nC, nE, nF.

Because A : B : : E : F, and because also A : B : : $mA : mB$ (V. 15), and E : F : : $nE : nF$; therefore, $mA : mB :: nE : nF$ (V. 11). Again, because B : C : : D : E, $mB : nC :: mD : nE$ (V. 4); and it has been just shown that $mA : mB :: nE : nF$; therefore, if $mA > nC$, $mD > nF$ (V. 21); if $mA = nC$, $mD = nF$; and if $mA < nC$, $mD < nF$. Now, mA and mD are any equimultiples of A and D, and nC, nF, any equimultiples of C and F; therefore, A : C : : D : F (V. Def. 10).

A	B,	C,
D,	E,	F.
mA,	mB,	nC,
mD,	nE,	nF.

Next, let there be four magnitudes A, B, C, and D, and other four, E, F, G, and H, which, taken two and two in a cross order, have the same ratio, namely, A : B : : G : H, B : C : : F : G, and C : D : : E : F, then A : D : : E : H. For, since A, B, C, are three magnitudes, and F, G, H, other three, which, taken two and two in a cross order, have the same ratio, by the first case A : C : : F : H. But C : D : : E : F; therefore again, by the first case, A : D : : E : H. In the same manner may the demonstration be extended to any number of magnitudes.

A,	B,	C,	D.
E,	F,	G,	H.

PROPOSITION XXIV. THEOREM.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second, the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let $A : B :: C : D$, and also $E : B :: F : D$, then $A + E : B :: C + F : D$.

Because $E : B :: F : D$, by inversion, $B : E :: D : F$. But by hypothesis, $A : B :: C : D$; therefore, by equality (V. 22), $A : E :: C : F$, and by composition (V. 18), $A + E : E :: C + F : F$. Now, again, by hypothesis, $E : B :: F : D$; therefore, by equality, $A + E : B :: C + F : D$.

PROPOSITION E. THEOREM.

Ratios which are compounded of the same ratios, are the same with one another.

Let the ratios of A to B, and of B to C, which compound the ratio of A to C, be equal, each to each, to the ratios of D to E, and E to F, which compound the ratio of E to F, $A : C :: D : F$.

For, first, if the ratio of A to B be equal to that of D to E, and the ratio of B to C equal to that of E to F, by equality (V. 22), $A : C :: D : F$.

A, B, C,
D, E, F.

And next, if the ratio of A to B be equal to that of E to F, and the ratio of B to C equal to that of D to E, by indirect equality (V. 23), $A : C :: D : F$. In the same manner may the proposition be demonstrated whatever be the number of ratios.

PROPOSITION F.

The terms of an analogy are proportional by addition.

If $A : B :: C : D$, then by addition $A : A+B :: C : C+D$.

For $B : A :: D : C$, by inversion (V. A); therefore $A+B : A :: C + D : C$ by composition (V. 18); hence, by inversion $A : A + B = C : C + D$ (V. A).

PROPOSITION G.

The terms of an analogy are proportional by mixing.

If $A : B :: C : D$, then $A + B : A - B :: C + D : C - D$.

For $A + B : B = C + D : D$ by composition (V. 18);

and by division $A - B : B :: C - D : D$ (V. 17), also by inversion (V. A)

$$B : A - B :: D : C - D;$$

$$\text{but } A + B : B :: C + D : D;$$

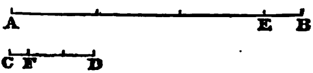
therefore, by equality (V. 22), $A + B : A - B :: C + D : C - D$.

If $B > A$, it may be similarly proved that $A + B : B - A :: C + D : D - C$.

PROPOSITION H. PROBLEM.

To find a common measure of two lines.

Let AB, CD, be the two lines.



Find the number of times that CD is contained in AB. If it be contained an exact number of times, then it is a measure of it, and any part of it will also be a common measure. But if it be contained several times in AB, as 3 times, with a remainder EB; then *if* EB be a measure of CD, it will also be one of AE, which is a multiple of CD; and therefore it will be also a measure of AB; it would therefore be the common measure required. But if EB be not a measure of CD, let it be contained in it a certain number of times, as 2 times, with the remainder CF; then if CF be a measure of EB, it will also be a measure of DF, which is a multiple of EB; and therefore it will also be a measure of CD, and consequently of AE, and therefore also of AB. Let CF be contained 2 times in EB, then $EB = 2CF$; $CD = 2EB + CF = 4CF + CF = 5CF$; and $AB = 3CD + EB = 15CF + 2CF = 17CF$. And CF is therefore contained 5 times in CD, and 17 times in AB.

In the same manner, the common measure of any other two commensurable lines may be found.

COR. 1.—If the process for finding a common measure of two lines never terminates, the lines are incommensurable.

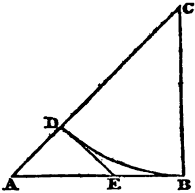
COR. 2.—Any part of a common measure is also a common measure; and the measure found as above, is the greatest common measure.

Cor. 3.—Any two commensurable lines are to one another, as the numbers denoting the number of times that they respectively contain their common measure.

PROPOSITION K. THEOREM.

The diagonal and side of a square are incommensurable.

Let ABC be the half of a square, the side AB and diagonal AC are incommensurable.



Since the angle at B is a right angle, each of the equal angles at C and A is less than a right angle (I. 32); therefore $AB < AC$. Again, $AB + BC > AC$ (I. 20), or $2AB > AC$; hence AC is greater than once AB, and less than twice AB; and the same may be proved of the diagonal and side of any square. Therefore, when the side of a square is taken once from its diagonal, there is always a remainder less than its side.

From C as a centre, with the radius CB, describe the arc DB; then $CD = CB = AB$, and $AD < AB$. From D draw DE perpendicular to AC; then ED and EB being tangents (III. 16, Cor.), $ED = EB$ (III. 37). But in the triangle ADE, the angle at D is a right angle, and that at A is half a right angle; therefore that at E is half a right angle, and therefore $AD = DE = EB$. When the first remainder AD therefore is taken from AB, the remaining part AE, from which AD is still to be taken, is the diagonal of a square, of which AD is the side. But this is the same as the former process; and when it has been performed in regard to AE and AD, the remaining lines to be compared will, therefore, again be the side and diagonal of a still smaller square; but since, when the side of a square is taken from its diagonal, there is a remainder, therefore in the above process there will always be found to be a remainder; the process therefore will never terminate, or no common measure can ever be found; therefore AC and AB are incommensurable.

EXERCISES.

1. If all the terms, or any two homologous terms, or the terms of either of the ratios, of an analogy, be multiplied or divided by the same number, the resulting magnitudes are still proportional.

2. If any number of magnitudes be in continued proportion, the difference between the first and second terms is to the first, as the difference between the first and last to the sum of all the terms except the last.

3. If the same magnitude be added to the terms of a ratio, it will be unchanged, increased, or diminished, according as it is a ratio of equality, minority, or majority.

4. The differences of the successive terms of continued proportionals are also in continued proportion.

5. The first term of an infinite decreasing series of quantities in continued proportion, is a mean proportional between its excess above the second, and the sum of the series.

SIXTH BOOK.

DEFINITIONS.

1. Straight lines are said to be *similarly divided*, when their corresponding segments are proportional.

2. A line is said to be cut in a *given ratio*, when its segments have that ratio.

3. Straight lines that meet in the same point, are called *convergent* or *divergent* lines, according as they are considered to verge towards or from the point of concourse.

4. A straight line is said to be cut in *extreme* and *mean ratio*, when the whole is to the greater segment, as the greater segment is to the less.

5. A straight line is said to be *harmonically* divided, when it is divided into three segments such, that the whole line is to one extreme segment, as the other extreme segment to the middle segment.

6. Three straight lines are said to be in *harmonical pro-*

gression or *harmonical progressionals*, when the first is to the third, as the difference between the first and second is to the difference between the second and third.

7. The second of three lines in this progression, is called a *harmonic mean* between the other two; and the third is called a *third harmonical progressional* to the other two.

8. Four divergent lines that cut any line harmonically, are called *harmonicals*.

9. *Similar rectilineal figures* are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

10. Two sides of one figure are said to be *reciprocally proportional* to two sides of another, when one of the sides of the first is to one of the sides of the other, as the remaining side of the other is to the remaining side of the first.

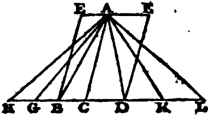
11. A *lune* is a figure contained by two arcs of different circles.

PROPOSITION I. THEOREM.

Triangles and parallelograms of the same altitude are one to another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF, have the same altitude, namely, the perpendicular drawn from the point A to BD; then, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of straight lines BG, GH, each equal to the base BC; and DK, KL, any number of them, each equal to the base CD; and join AG, AH, AK, AL; then, because CB, BG, GH, are all equal, the triangles AHG, AGB, ABC, are all equal (I. 38); therefore, whatever multiple the base HC is of the base BC, the same multiple is the triangle AHC of the triangle ABC. For the same reason, whatever multiple the base LC is of the base CD, the same multiple is the triangle ALC of the triangle ADC. And if the base HC be equal to the base CL, the triangle AHC is also equal to the tri-



angle ALC; and if the base HC be greater than the base CL, likewise the triangle AHC is greater than the triangle ALC; and if less, less. Therefore, since there are four magnitudes, namely, the two bases BC, CD, and the two triangles ABC, ACD; and of the base BC and the triangle ABC, the first and third, any equimultiples *whatever have* been taken, namely, the base HC and triangle AHC; and of the base CD and triangle ACD, the second and fourth, have been taken any equimultiples whatever, namely, the base CL and triangle ALC; and since it has been shown, that, if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if less, less. Therefore (V. Def. 10), as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

And because the parallelogram CE is double of the triangle ABC (I. 41), and the parallelogram CF double of the triangle ACD, and that magnitudes have the same ratio which their equimultiples have (V. 15); as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF. And because it has been shown, that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD; and as the triangle ABC to the triangle ACD, so is the parallelogram EC to the parallelogram CF; therefore, as the base BC is to the base CD, so is the parallelogram EC to the parallelogram CF (V. 11).

Cor. 1.—From this it is plain, that triangles and parallelograms that have equal altitudes, are one to another as their bases.

Let the figures be placed so as to have their bases in the same straight line, then the straight line which joins the vertices is parallel to that in which their bases are (I. 34, Cor. 2). Then, if the same construction be made as in the proposition, the demonstration will be the same.

Cor. 2.—Rectangles, and hence also parallelograms and triangles, having equal bases, are to one another as their altitudes.

PROPOSITION II. THEOREM.

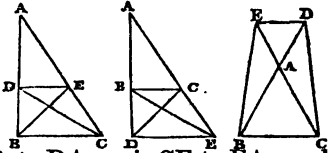
If a straight line be drawn parallel to one of the sides of

a triangle, it shall cut the other sides, or the other sides produced, proportionally; and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Let DE be drawn parallel to BC, one of the sides of the triangle ABC; BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle CDE (I. 37), because they are on the same base DE, and between the same parallels DE, BC. But ADE is another triangle, and equal magnitudes have to the same the same ratio (V. 7); therefore, as the triangle BDE to the triangle ADE, so is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, so is BD to DA (VI. 1); because, having the same altitude, namely, the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA. Therefore, as BD to DA, so is CE to EA (V. 11).

Next, let the sides AB, AC, of the triangle ABC, or these sides produced, be cut proportionally in the points D, E, that is, so that BD be to DA, as CE to EA, and join DE; DE is parallel to BC.



The same construction being made, because as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE (VI. 1); and as CE to EA, so is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE, have the same ratio to the triangle ADE; and therefore the triangle BDE is equal to the triangle CDE (V. 9). And they are on the same base DE; but equal triangles on the same base are between the same parallels (I. 39); therefore DE is parallel to BC.

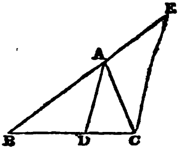
PROPOSITION III. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall

have the same ratio which the other sides of the triangle have to one another; and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section, bisects the vertical angle.

Let the angle BAC of any triangle ABC be divided into two equal angles by the straight line AD; BD is to DC, as BA to AC.

Through the point C draw CE parallel to DA (I. 31), and let BA produced meet CE in E. Because the straight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD (I. 29). But CAD, by the hypothesis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, because the straight line BAE meets the parallels AD, EC, the exterior angle BAD is equal to the interior and opposite angle AEC. But the angle ACE has been proved equal to the angle *BAD*; therefore also ACE is equal to the angle AEC, and consequently the side AE is equal to the side AC (I. 6). And because AD is drawn parallel to one of the sides of the triangle BCE, namely, to EC, BD is to DC, as BA to AE (VI. 2). But AE is equal to AC; therefore, as BD to DC, so is BA to AC (V. 7.)



Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the straight line AD.

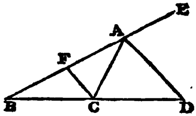
The same construction being made; because, as BD to DC, so is BA to AC; and as BD to DC, so is BA to AE (VI. 2), because AD is parallel to EC; therefore AB is to AC, as BA to AE (V. 11); consequently AC is equal to AE (V. 9), and the angle AEC is therefore equal to the angle ACE (I. 5). But the angle AEC is equal to the external and opposite angle BAD; and the angle ACE is equal to the alternate angle CAD. Wherefore, also, the angle BAD is equal to the angle CAD; therefore the angle BAC is cut into two equal angles by the straight line AD.

PROPOSITION A. THEOREM.

If the exterior angle of a triangle be bisected by a straight line which also cuts the base produced, the segments between the bisecting line and the extremities of the base have the same ratio which the other sides of the triangle have to one another; and if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section bisects the exterior angle of the triangle.

Let the exterior angle CAE of any triangle ABC be divided into two equal angles by the straight line AD which meets the base produced in D; BD is to DC, as BA to AC.

Through C draw CF parallel to AD (I. 31); and because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (I. 29); but CAD is equal to the angle DAE (Hyp.); therefore also DAE is equal to the angle ACF. Again, because the straight line FAE meets the parallels AD, FC, the exterior angle DAE is equal to the interior and opposite angle CFA. But the angle ACF has been proved to be equal to the angle DAE; therefore also the angle ACF is equal to the angle CFA; and consequently the side AF is equal to the side AC (I. 6); and because AD is parallel to FC, a side of the triangle BCF; BD is to DC, as BA to AF (VI. 2); but AF is equal to AC; as therefore BD is to DC, so is BA to AC.



Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

The same construction being made, because BD is to DC, as BA to AC; and also BD to DC, as BA to AF; therefore BA is to AC, as BA to AF (V. 11); wherefore AC is equal to AF (V. 9), and the angle AFC equal to the angle ACF (I. 5). But the angle AFC is equal to the exterior angle EAD, and the angle ACF to the alternate angle CAD; therefore also EAD is equal to the angle CAD.

Cor.—The two lines that bisect the vertical angle and its adjacent angle, cut the base produced harmonically; or the base is a harmonic mean between its greater internal and external segments.

Schol. 1.—This and the last propositions may be enunciated thus:—If the vertical angle of a triangle and its adjacent angle be bisected by lines cutting the base, it will be cut internally and externally in the ratio of the two sides.

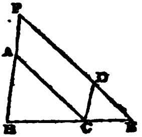
Schol. 2.—By means of these two propositions, it is proved in optics, that the axis of a pencil of rays incident on a spherical mirror is divided harmonically by the radiant point, the geometrical focus of reflected rays, and the centre and surface of the reflector. It is also found that the lengths of three musical strings of the same thickness, material, and texture, and under the same tension, that produce any note, its fifth, and octave, are in harmonical progression; and hence the origin of the term. It is believed that Pythagoras first observed this relation of musical strings.

PROPOSITION IV. THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE, be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and consequently the angle BAC equal to the angle CDE (I. 32). The sides about the equal angles of the triangles ABC, DCE, are proportionals; and these are the homologous sides which are opposite to the equal angles.

Let the triangle DCE be placed, so that its side CE may be contiguous to BC, and in the same straight line with it; and because the angles ABC, ACB, are together less than two right angles (I. 17), ABC and DEC, which is equal to ACB, are also less than two right angles; wherefore BA, ED, produced, shall meet (I. 29, Cor.); let them be produced and meet in the point F; and because the angle ABC is equal to the angle DCE, BF is parallel to CD (I. 28). Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE; therefore FACD is a parallelogram; and consequently AF

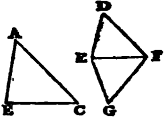


is equal to CD, and AC to FD (I. 34); and because AC is parallel to FE, one of the sides of the triangle FBE, BA : AF : : BC : CE (VI. 2); but AF is equal to CD; therefore (V. 7), BA : CD : : BC : CE; and alternately, AB : BC : : DC : CE. Again, because CD is parallel to BF, BC : CE : : FD : DE; but FD is equal to AC; therefore, BC : CE : : AC : DE; and alternately, BC : CA : : CE : ED. Therefore, because it has been proved that AB : BC : : DC : CE; and BC : CA : : CE : ED, by equality, BA : AC : : CD : DE.

PROPOSITION V. THEOREM.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let the triangles ABC, DEF, have their sides proportionals, so that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and consequently, by equality, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides; namely, the angle ABC being equal to the angle DEF, and BCA to EFD, and also BAC to EDF.



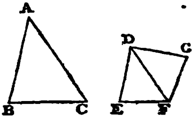
At the points E, F, in the straight line EF, make (I. 23) the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle EGF (I. 32), and the triangle ABC is therefore equiangular to the triangle GEF; and consequently they have their sides opposite to the equal angles proportionals (VI. 4); wherefore, as AB to BC, so is GE to EF; but as AB to BC, so is DE to EF; therefore, as DE to EF, so GE to EF (V. 11); therefore DE and GE have the same ratio to EF, and consequently are equal (V. 9); for the same reason, DF is equal to FG; and because, in the triangles DEF, GEF, DE is equal to EG, and EF common, and also the base DF equal to the base GF; therefore (I. 8) the angle DEF is equal to the angle GEF, DFE to GFE, and EDF to EGF; and because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF; for the same reason, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D; therefore the triangle ABC is equiangular to the triangle DEF.

PROPOSITION VI. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF, have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC, as ED to DF; the triangles ABC, DEF, are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

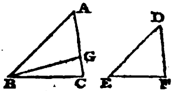
At the points D, F, in the straight line DF, make *(I. 23)* the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB; wherefore the remaining angle at B is equal to the remaining one at G (I. 32), and consequently the triangle ABC is equiangular to the triangle DGF; and therefore as BA to AC, so is GD to DF (VI. 4); but, by the hypothesis, as BA to AC, so is ED to DF; as therefore ED to DF, so is GD to DF (V. 11); wherefore ED is equal to DG (V. 9); and DF is common to the two triangles EDF, GDF; therefore the two sides ED, DF, are equal to the two sides GD, DF; but the angle EDF is also equal to the angle GDF; therefore (I. 4) the angle DFG is equal to the angle DFE, and the angle at G to the angle at E. But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE; and the angle BAC is equal to the angle EDF (Hyp.); wherefore also the remaining angle at B is equal to the remaining angle at E; therefore the triangle ABC is equiangular to the triangle DEF.



PROPOSITION VII. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then, if each of the remaining angles be either less, or not less, than a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.

Let the two triangles ABC, DEF, have one angle in the one equal to one angle in the other; namely, the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF, proportionals, so that AB is to BC, as DE to EF; and in the first case, let each of the remaining angles at C, F, be less than a right angle. The triangle ABC is equiangular to the triangle DEF; that is, the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

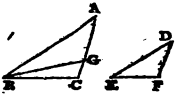


For, if the angles ABC, DEF, be not equal, one of them is greater than the other. Let ABC be the greater, and at the point B, in the straight line AB, make the angle ABG equal to the angle DEF (I. 23); and because the angle at A is equal to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal to the remaining angle DFE (I. 32); therefore the triangle ABG is equiangular to the triangle DEF; wherefore (VI. 4), as AB is to BG, so is DE to EF; but as DE to EF, so, by hypothesis, is AB to BC; therefore, as AB to BC, so is AB to BG (V. 11); and because AB has the same ratio to each of the lines BC, BG; BC is equal to BG (V. 9), and therefore the angle BGC is equal to the angle BCG (I. 6); but the angle BCG is, by hypothesis, less than a right angle; therefore also the angle BGC is less than a right angle, and the adjacent angle AGB must be greater than a right angle (I. 13). But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle; but, by the hypothesis, it is less than a right angle; which is absurd; therefore the angles ABC, DEF, are not unequal, that is, they are equal; and the angle at

A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F; therefore the triangle ABC is equiangular to the triangle DEF.

Next, let each of the angles at C, F, be not less than a right angle; the triangle ABC is also in this case equiangular to the triangle DEF.

The same construction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC; but the angle at C is not less than a right angle; therefore the angle BGC is not less than a right angle; wherefore, two angles of the triangle BGC are together not less than two right angles, which is impossible (I. 17); and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.

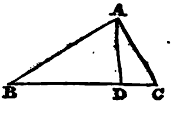


PROPOSITION VIII. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC; the triangles ABD, ADC, are similar to the whole triangle ABC, and to one another.

Because the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD (I. 32); therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals (VI. 4); wherefore the triangles are similar (VI. Def. 9); in the same manner, it may be demonstrated that the triangle ADC is equiangular and similar to the triangle ABC; and the triangles ABD, ADC, being both equiangular and similar to ABC, are equiangular and similar to each other.



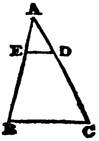
Cor.—From this it is manifest that the perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base; and also that each of the sides is a mean proportional between the base, and its segment adjacent to that side. For in the triangles BDA, ADC, BD : DA : : DA : DC (VI. 4); and in the triangles ABC, DBA, BC : BA : : BA : BD; and in the triangles ABC, ACD, BC : CA : : CA : CD.

PROPOSITION IX. PROBLEM.

From a given straight line to cut off any part required; that is, a part of which the line shall be a given multiple.

Let AB be the given straight line; it is required to cut off from it a part of which AB shall be a given multiple.

From the point A draw a straight line AC making any angle with AB; and in AC take any point D, and take AC the same multiple of AD, that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it; then AE is the part required to be cut off.



Because ED is parallel to one of the sides of the triangle ABC, namely, to BC, as CD is to DA, so is BE to EA (VI. 2); and, by composition (V. 18), CA is to AD, as BA to AE. But CA is a multiple of AD; therefore (V. c) BA is the same multiple of AE. Whatever part, therefore, AD is of AC, AE is the same part of AB; wherefore, from the straight line AB the part required is cut off.

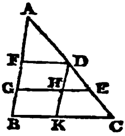
PROPOSITION X. PROBLEM.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Let AB be the straight line given to be divided, and AC the divided line; it is required to divide AB similarly to AC.

Let AC be divided in the points D, E; and let AB, AC, be placed so as to contain any angle, and join BC, and through the points D, E, draw DF, EG, parallel to BC (I. 31); and through D draw DHK parallel to AB; therefore each of the figures FH, HB, is a parallelogram; wherefore

DH is equal to FG, and HK to GB (I. 34). And because HE is parallel to KC, one of the sides of the triangle DKC, as CE to ED, so is KH to HD (VI. 2). But KH is equal to BG, and HD to GF; therefore, as CE to ED, so is BG to GF. Again, because FD is parallel to EG, one of the sides of the triangle AGE, as ED to DA, so is GF to FA. But it has been proved that CE is to ED, as BG to GF; and as ED to DA, so GF to FA; therefore the given straight line AB is divided similarly to AC.



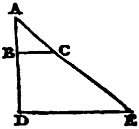
PROPOSITION XI. PROBLEM.

To find a third proportional to two given straight lines.

Let AB, AC, be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC; and make BD equal to AC; and having joined BC, through D, draw DE parallel to it (I. 31).

Because BC is parallel to DE, a side of the triangle ADE, AB is to BD, as AC to CE (VI. 2). But BD is equal to AC; as therefore AB to AC, so is AC to CE. Wherefore to the two given straight lines AB, AC, a third proportional CE is found.

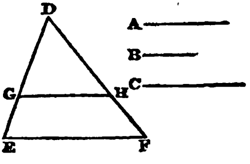


PROPOSITION XII. PROBLEM.

To find a fourth proportional to three given straight lines.

Let A, B, C, be the three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF, containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel to it through the point E (I. 31); and because GH is parallel to EF, one of the sides of the triangle DEF; DG is to GE, as DH to HF (VI. 2); but DG is equal to A,



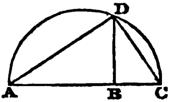
GE to B, and DH to C; therefore, as A is to B, so is C to HF. Wherefore to the three given straight lines, A, B, C, a fourth proportional HF is found.

PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given straight lines.

Let AB, BC, be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC, in a straight line, and upon AC describe the semicircle ADC, and (I. 11) from the point B draw BD at right angles at AC, and join AD, DC.

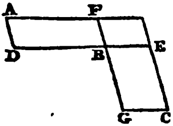


Because the angle ADC in a semicircle is a right angle (III. 31), and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB, BC, the segments of the base (VI. 8, Cor.); therefore between the two given straight lines AB, BC, a mean proportional DB is found.

PROPOSITION XIV. THEOREM.

Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC, be equal parallelograms, which have the angles at B equal, and let the sides DB, BE, be placed in the same straight line; wherefore also FB, BG, are in one straight line (I. 14); the sides of the parallelograms AB, BC, about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF.



Complete the parallelogram FE; and because the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE (V. 7). But as AB to FE, so is the base DB to BE (VI. 1); and, as BC to FE, so is the base GB to BF; therefore, as DB to BE, so is GB to BF (V. 11). Wherefore, the sides of the parallelograms AB, BC, about their equal angles are reciprocally proportional.

And, if the sides about the equal angles be reciprocally proportional, namely, as DB to BE, so GB to BF; the parallelogram AB is equal to the parallelogram BC.

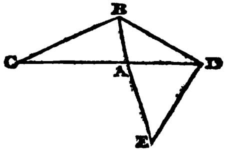
Because, as DB to BE, so is GB to BF; and as DB to BE, so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE; therefore, as AB to FE, so BC to FE. Wherefore the parallelogram AB is equal to the parallelogram BC (V. 9).

PROPOSITION XV. THEOREM.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE, be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD, be in one straight line; wherefore also EA and AB are in one straight line (I. 14); join BD. Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore, as the triangle CAB is to the triangle BAD, so is triangle EAD to triangle DAB (V. 7). But as triangle CAB to triangle BAD, so is the base CA to AD (VI. 1); and as triangle EAD to triangle DAB, so is the base EA to AB; as therefore CA to AD so is EA to AB (V. 11); wherefore the sides of the triangles ABC, ADE, about the equal angles are reciprocally proportional.



And, if the sides of the triangles ABC, ADE, about the equal angles be reciprocally proportional, namely CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

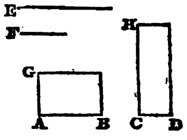
Having joined BD as before; because, as CA to AD, so is EA to AB; and as CA to AD, so is triangle ABC to triangle BAD (VI. 1); and as EA to AB, so is triangle EAD to triangle BAD; therefore (V. 11), as triangle BAC to triangle BAD, so is triangle EAD to triangle BAD; that is, the triangles BAC, EAD, have the same ratio to the triangle BAD. Wherefore the triangle BAC is equal to the triangle ADE (V. 9).

PROPOSITION XVI. THEOREM.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines AB, CD, E, F, be proportionals; namely, as AB to CD, so E to F; the rectangle contained by AB, F, is equal to the rectangle contained by CD, E.

From the points A, C, draw AG, CH, at right angles to AB, CD (I. 11); and make AG equal to F, and CH equal to E, and complete the parallelograms BG, DH; because, as AB to CD, so is E to F; and that E is equal to CH, and F to AG; AB is to CD, as CH to AG (V. 7); therefore the sides of the parallelograms BG, DH, about the equal angles, are reciprocally proportional; but parallelograms which have their sides about equal angles reciprocally proportional, are equal to one another (VI. 14); therefore the parallelogram BG is equal to the parallelogram DH; and the parallelogram BG is contained by the straight lines AB, F; because AG is equal to F; and the parallelogram DH is contained by CD and E; because CH is equal to E; therefore the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD and E.



And if the rectangle contained by the straight lines AB, F, be equal to that which is contained by CD, E; these four lines are proportionals, namely, AB is to CD, as E to F.

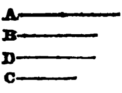
The same construction being made, because the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD, E, and that the rectangle BG *is contained* by AB, F, because AG is equal to F; and *the* rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular; but the sides about the equal angles of equal parallelograms are reciprocally proportional (VI. 14); wherefore, as AB to CD, so is CH to AG; and CH is equal to E, and AG to F; as therefore AB is to CD, so E to F.

PROPOSITION XVII. THEOREM.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean; and if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are *proportionals.*

Let the three straight lines A, B, C, be proportionals; namely, as A to B, so B to C; the rectangle contained by A, C, is equal to the square of B.

Take D equal to B; and because as A to B, so B to C, and that B is equal to D; A is to B, as D to C; but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means (VI. 16); therefore the rectangle contained by A, C, is equal to that contained by B, D; but the rectangle contained by B, D, is the square of B; because B is equal to D; therefore the rectangle contained by A, C, is equal to the square of B.



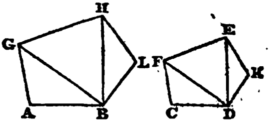
And if the rectangle contained by A, C, be equal to the square of B; A is to B, as B to C.

The same construction being made, because the rectangle contained by A, C, is equal to the square of B, and the square of B is equal to the rectangle contained by B, D, because B is equal to D; therefore the rectangle contained by A, C, is equal to that contained by B, D; but if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals; therefore A is to B, as D to C; but B is equal to D; wherefore as A to B, so B to C.

PROPOSITION XVIII. PROBLEM.

Upon a given straight line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides; it is required upon the given straight line AB to describe a rectilineal figure similar, and similarly situated, to CDEF.



Join DF, and at the points A, B, in the straight line AB, make (I. 23) the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefore the remaining angle CFD is equal to the remaining angle AGB (I. 32); wherefore the triangle FCD is equiangular to the triangle GAB. Again, at the points G, B, in the straight line GB, make (I. 23) the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH; then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE; for the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED; therefore the rectilineal figure ABHG is equiangular to CDEF; but likewise these figures have their sides about the equal angles proportionals; because the triangles GAB, FCD, being equiangular, BA is to AG, as DC to CF (VI. 4); and because AG is to GB, as CF to FD; and as GB to GH, so is FD to FE (VI. 4); therefore (V. 22) AG is to GH, as CF to FE; in the same manner, it may be proved that AB is to BH, as CD to DE; and GH is to HB, as FE to ED (VI. 4); wherefore, because the rectilineal figures ABHG, CDEF, are equiangular, and have their sides about the equal angles proportionals, they are similar to one another (VI. Def. 9).

Next, let it be required to describe upon a given straight line AB, a rectilineal figure similar, and similarly situated, to the rectilineal figure CDKEF.

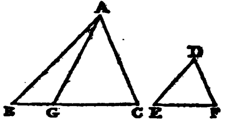
Join DE, and upon the given straight line *AB describe* the rectilineal figure ABHG similar, and similarly situated, to the quadrilateral figure CDEF, by the former case; and at the points B, H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L; and because the figures ABHG, CDEF, are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK; for the same reason, the angle ABL is equal to the angle CDK; therefore the five-sided figures AGHLB, CFEKD, are equiangular; and because the figures AGHB, CFED, are similar, GH is to HB, as FE to ED; and as *HB to HL*, so is ED to EK (VI. 4); therefore (V. 22) GH *is to HL*, as FE to EK; for the same reason, AB is to BL, as CD to DK; and BL is to LH, as DK to KE, because the triangles BLH, DKE, are equiangular; therefore, because the five-sided figures AGHLB, CFEKD, are equiangular, and have their sides about the equal angles proportionals, they are similar to one another; and in the same manner, a rectilineal figure of six or more sides may be described upon a given straight line similar to one given; and so on.

PROPOSITION XIX. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF, be similar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, so that the side BC is homologous to EF (V. Def. 20); the triangle ABC has to the triangle DEF, the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF (VI. 11), so that BC is to EF, as EF to BG, and join GA; then, be-



cause as AB to BC, so DE to EF; alternately (V. 16), AB is to DE, as BC to EF. But as BC to EF, so is EF to BG; therefore (V. 11) as AB to DE, so is EF to BG. Wherefore the sides of the triangles ABG, DEF, which are about the equal angles, are reciprocally proportional. But triangles which have the sides about two equal angles reciprocally proportional are equal to one another (VI. 15); therefore the triangle ABG is equal to the triangle DEF; and because as BC is to EF, so EF to BG; and that if three straight lines be proportionals, the first has to the third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF. But as BC to BG, so is the triangle ABC to the triangle ABG (VI. 1). Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF. But the triangle ABG is equal to the triangle DEF; wherefore also the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF.

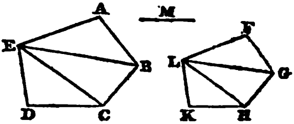
COR.—From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar, and similarly described triangle upon the second.

PROPOSITION XX. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL, be similar polygons, and let AB be the homologous side to FG; the polygons ABCDE, FGHKL, may be divided into the same number of similar triangles, whereof each to each has the same ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the side FG.

Join BE, EC, GL, LH; and because the plygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL; and BA is to AE, as GF to FL (VI. Def. 9); wherefore, because the triangles ABE, FGL, have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle ABE is equiangular (VI. 6), and therefore similar to the triangle FGL (VI. 4); wherefore the angle ABE is equal to the angle FGL. And because the polygons are similar, the whole angle ABC is equal to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH. And because the triangles ABE, FGL, are similar, EB is to BA, as LG to GF. And also, because the polygons are similar, AB is to BC, as FG to GH; therefore (V. 22) EB is to BC, as LG to GH; that is, the sides about the equal angles EBC, LGH, are proportionals; therefore the triangle EBC is equiangular to the triangle LGH, and similar to it. For the same reason, the triangle ECD likewise is similar to the triangle LHK; therefore the similar polygons ABCDE, FGHKL, are divided into the same number of similar triangles.



Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK. And the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio of that which the side BE has to the side GL (VI. 19). For the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL; therefore, as the triangle ABE to the triangle FGL, so is the triangle BEC to the triangle GLH (V. 11). Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH. For the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH; as therefore the triangle EBC to the triangle LGH, so is the triangle ECD to the triangle LHK. But it has been proved, that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to the triangle FGL. Therefore, as the triangle ABE is to the triangle FGL, so is triangle EBC to triangle LGH, and triangle ECD to triangle LHK; and therefore, as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents (V. 12). Wherefore, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL; but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the side AB has to the homologous side FG. Therefore, also, the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG.

Cor. 1.—In like manner it may be proved, that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and it has already been proved in triangles. Therefore, universally similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Cor. 2.—If to AB, FG, two of the homologous sides, a third proportional M be taken, AB has (V. Def. 18) to M the duplicate ratio of that which AB has to FG. But the four-sided figure or polygon upon AB has to the four-sided figure or polygon upon FG likewise the duplicate ratio of that which AB has to FG; therefore, as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in triangles (VI. 19, Cor.) Therefore, universally, it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

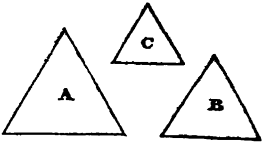
Cor. 3.—Because all squares are similar figures, the ratio of any two squares to one another is the same with the duplicate ratio of their sides; and hence, also, any two similar rectilineal figures are to one another as the squares of their homologous sides.

PROPOSITION XXI. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Let each of the rectilineal figures A, B, be similar to the rectilineal figure C; the figure A is similar to the figure B.

Because A is similar to C, they are equiangular, and also have their sides about the equal angles proportionals (VI. Def. 9). Again, because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals; therefore the figures A, B, are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals; wherefore the rectilineal figures A and B are equiangular, and have their sides about the equal angles proportionals (V. 11); therefore A is similar to B.

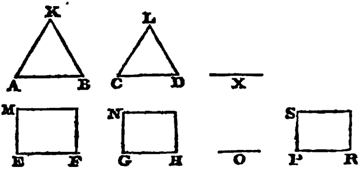


PROPOSITION XXII. THEOREM.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, GH, be proportionals; namely, AB to CD, as EF to GH, and upon AB, CD, let the similar rectilineal figures KAB, LCD, be similarly described; and upon EF, GH, the similar rectilineal figures MF, NH, in like manner; the rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD, take a third proportional X (VI. 11); and to EF, GH, a third proportional O; and because AB is to CD, as EF to GH, and that CD is to X, as GH to O (V. 11); wherefore (V. 22) as AB to X, so EF to O; but as AB to X, so is the rectilineal KAB to the rec-



tilineal LCD (VI. 20, Cor. 2), and as EF to O, so is the rectilineal MF to the rectilineal NH; therefore, as KAB to LCD, so is MF to NH.

And if the rectilineal KAB be to LCD, as MF to NH; the straight line AB is to CD, as EF to GH.

Make (VI. 12) as AB to CD, so EF to PR, and (VI. 18) upon PR describe the rectilineal figure SR similar and similarly situated to either of the figures MF, NH; then, because as AB to CD, so is EF to PR, and that upon AB, CD, are described the similar and similarly situated rectilineals KAB, LCD, and upon EF, PR, in like manner, the similar rectilineals MF, SR; KAB is to LCD, as MF to SR; but, by the hypothesis, KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the same ratio to each of the two NH, SR, these are equal to one another (V. 9); they are also similar, and similarly situated; therefore GH is equal to PR; and because as AB to CD, so is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH.

Cor.—If four straight lines be proportional, their squares are proportional, and conversely.

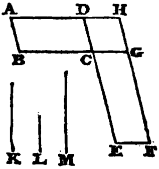
For squares are similar figures.

PROPOSITION XXIII. THEOREM.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Let AC, CF, be equiangular parallelograms, having the angle BCD equal to the angle ECG; the ratio of the parallelogram AC to the parallelogram CF, is the same with the ratio which is compounded of the ratios of their sides.

Let BC, CG, be placed in a straight line; therefore DC and CE are also in a straight line (I. 14); and complete the parallelogram DG; and, taking any straight line K, make as BC to CG, so K to L (VI. 12); and as DC to CE, so make L to M; therefore the ratios of K to L, and L to M, are the same with the ratios of the sides; namely, of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded of the ratios of K to L, and L to



M (V. Def. 17); wherefore also K has to M the ratio compounded of the ratios of the sides; and because as BC to CG, so is the parallelogram AC to the parallelogram CH (VI. 1); but as BC to CG, so is K to L; therefore K is to L, as the parallelogram AC to the parallelogram CH (V. 11); again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; but as DC to CE, so is L to M; wherefore L is to M, as the parallelogram CH to the parallelogram CF; therefore, since it has been proved that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so the parallelogram CH to the parallelogram CF; K is to M, as the parallelogram AC to the parallelogram CF (V. 22); but K has to M the ratio which is compounded of the ratios of the sides; therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides.

Cor. 1.—The rectangles under the corresponding terms of two analogies are proportional.

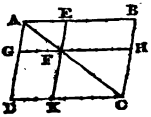
For let $A : B = C : D$ and $E : F = G : H$, and let M, N, P, Q, denote respectively the rectangles $A \cdot E$, $B \cdot F$, $C \cdot G$, and $D \cdot H$; then $M : N$ in a ratio compounded of $A : B$ and $E : F$ (VI. 23), and $P : Q$ in a ratio compounded of $C : D$ and $G : H$; but the two former simple ratios are respectively equal to the two latter, and therefore the ratios compounded of them are equal (V. E); therefore $M : N = P : Q$.

Cor. 2.—Hence rectangles, whose bases are proportional, and also their altitudes, are themselves proportional.

PROPOSITION XXIV. THEOREM.

The parallelograms about the diameter of any parallelogram, are similar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK, the parallelograms about the diameter; the parallelograms EG, HK, are similar both to the whole parallelogram ABCD, and to one another.



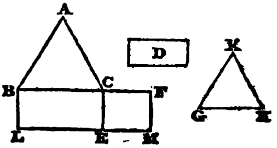
Because DC, GF, are parallels, the angle ADC is equal to the angle AGF (I. 29); for the same reason, because BC, EF, are parallels, the angle ABC is equal to the angle AEF; and each of the angles BCD, EFG, is equal to the opposite angle DAB (I. 34), and therefore are equal to one another, wherefore the parallelograms ABCD, AEFG, are equiangular; and because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore (VI. 4) as AB to BC, so is AE to EF; and because the opposite sides of parallelograms are equal to one another, AB is to AD, as AE to AG (V. 7); and DC to CB, as GF to FE; and also CD to DA, as FG to GA; therefore the sides of the parallelograms ABCD, AEFG, about the equal angles are proportionals; and they are therefore similar to one another (VI. Def. 9); for the same reason, the parallelogram ABCD is similar to the parallelogram FHCK; wherefore each of the parallelograms GE, KH, is similar to DB; but rectilineal figures which are similar to the same rectilineal figure, are also similar to one another (VI. 21); therefore the parallelogram GE is similar to KH.

PROPOSITION XXV. PROBLEM.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be described is required to be similar, and D that to which it must be equal. It is required to describe a rectilineal figure similar to ABC, and equal to D.

Upon the straight line BC describe the parallelogram BE equal to the figure ABC (I. 45, Cor.); also upon CE describe the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL; therefore BC and CF are in a straight line (I. 14), as also LE and EM (I. 29); between BC and CF find a mean proportional GH (VI. 13), and upon GH describe the rectilineal figure KGH similar and similarly situated to the figure ABC (VI. 18). And because BC is to GH as GH to CF, and if three straight lines be proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the



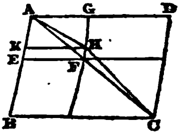
second (VI. 20, Cor. 2); therefore, as BC to CF, so is the rectilineal figure ABC to KGH. But as BC to CF, so is the parallelogram BE to the parallelogram EF (VI. 1); therefore, as the rectilineal figure ABC is to KGH, so is the parallelogram BE to the parallelogram EF (V. 11). And the rectilineal figure ABC is equal to the parallelogram BE; therefore the rectilineal figure KGH is equal to the parallelogram EF (V. 14). But EF is equal to the figure D; wherefore also KGH is equal to D; and it is similar to ABC. Therefore the rectilineal figure KGH has been described similar to the figure ABC, and equal to D.

PROPOSITION XXVI. THEOREM.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the parallelograms ABCD, AEFG, be similar and similarly situated, and have the angle DAB common; ABCD and AEFG are about the same diameter.

For, if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC; therefore the parallelograms ABCD, AKHG, being about the same diameter, they are similar to one another (VI. 24); wherefore, as DA to AB, so is (VI. Def. 9) GA to AK. But because ABCD and AEFG are similar parallelograms, as DA is to AB, so is GA to AE; therefore (V. 11) as GA to AE, so GA to AK; wherefore GA has the same ratio to each of the straight lines AE, AK; and consequently (V. 9) AK is equal to AE, the less to the greater, which is impossible; therefore ABCD and AKHG are not about the same diameter; wherefore ABCD and AEFG must be about the same diameter.

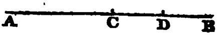


PROPOSITION XXVII. THEOREM.

Of all the rectangles contained by the segments of a given straight line, the greatest is the square which is described on half the line.

Let AB be a given straight line, which is bisected in C; and let D be any point in it, the square on AC is greater than the rectangle AD·DB.

For since the straight line AB is divided into two equal parts in C, and into two unequal parts in D, the rectangle contained by AD and DB, together with the square of CD, is equal to the square of AC (II. 5). The square of AC is therefore greater than the rectangle AD·DB.

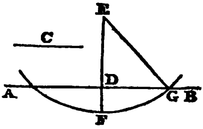


PROPOSITION XXVIII. PROBLEM.

To divide a given straight line, so that the rectangle contained by its segments may be equal to a given space; but that space must not be greater than the square of half the given line.

Let AB be the given straight line, and let the square upon the given straight line C be the space to which the rectangle contained by the segments of AB must be equal, and this square, by the determination, is not greater than that upon half the straight line AB.

Bisect AB in D, and if the square upon AD be equal to the square upon C, the thing required is done. But if it be not equal to it, AD must be greater than C, according to the determination. Draw DE at right angles to AB, and make it equal to C; produce ED to F, so that EF be equal to AD or DB, and from the centre E, at the distance EF, describe a circle meeting AB in G. Join EG; and because AB is divided equally in D, and unequally in G, the rectangle AG·GB, together with the square of DG, is equal to the square of DB (II. 5), that is, of EF or EG; but the squares of ED, DG, are also equal to the square of EG (I. 47); therefore the rectangle AG·GB, together with the square of DG, is equal to the squares of ED and DG. Take away the square of DG from each of these equals; therefore the remaining rectangle AG·GB is equal to the square of ED, that is, of C. Wherefore the given line AB is divided in G, so that the rectangle contained by the segments AG, GB, is equal to the square upon the given straight line C.

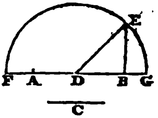


PROPOSITION XXIX. PROBLEM.

To produce a given straight line, so that the rectangle contained by the external segments of the given line may be equal to a given space.

Let AB be the given straight line, and let the square upon the given straight line C be the space to which the rectangle under the segments of AB produced, must be equal.

Bisect AB in D, and draw BE at right angles to it, so that BE be equal to C; and having joined DE, from the centre D at the distance DE describe a circle meeting AB produced in G. And because AB is bisected in D, and produced to G, the rectangle AG · GB, together with the square of DB, is equal to the square of DG (II. 6), or of DE, that is, to the squares of EB, BD (I. 47). From each of these equals take the square of DB; therefore the remaining rectangle AG · GB is equal to the square of BE, that is, to the square upon C. Wherefore the line AB is produced to G, so that the rectangle contained by the segments AG, GB, of the line produced, is equal to the square of C.

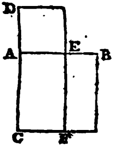


PROPOSITION XXX. PROBLEM.

To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

Upon AB describe the square BC (I. 46), and produce CA to D, so that the rectangle CD · DA may be equal to the square CB (VI. 29). Take AE equal to AD, and complete the rectangle DF under DC and AE, or under DC and DA. Then, because the rectangle CD, DA, is equal to the square CB, the rectangle DF is equal to CB. Take away the common part CE from each, and the remainder FB is equal to the remainder DE. But FB is the rectangle contained by FE and EB, that is, by AB and BE; and DE is the square upon AE; therefore AE is a mean proportional between AB and BE (VI. 17); or



AB is to AE as AE to EB. But AB is greater than AE; wherefore AE is greater than EB (V. 14); therefore the straight line AB is cut in extreme and mean ratio in E (VI. Def. 4).

Otherwise, let AB be the given straight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point C, so that the rectangle contained by AB, BC, be equal to the square of AC (II. 11); then, because the rectangle AB · BC is equal to the square of AC, as BA to AC, so is AC to CB (VI. 17); therefore AB is cut in extreme and mean ratio in C.

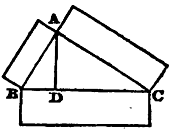
Cor.—The section of a line in extreme and mean ratio, is the same as medial section (II. Def. 6).

PROPOSITION XXXI. THEOREM.

In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

Let ABC be a right-angled triangle, having the right angle BAC; the rectilineal figure described upon BC is equal to the similar and similarly described figures upon BA, AC.

Draw the perpendicular AD; therefore, because in the right-angled triangle ABC, AD is drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC, are similar to the whole triangle ABC, and to one another (VI. 8); and because the triangle ABC is similar to ADB, as CB to BA, so is BA to BD (VI. 4); and because these three straight lines are proportionals, as the first to the third, so is the figure upon the first to the similar and similarly described figure upon the second (VI. 20, Cor. 2). Therefore, as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA; and inversely (V. A), as DB to BC, so is the figure upon BA to that upon BC; for the same reason, as DC to CB, so is the figure upon CA to that upon CB. Wherefore, as BD and DC together to BC, so are the

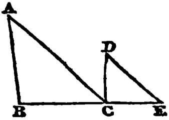


figures upon BA, AC, to that upon BC (V. 24); but BD and DC together are equal to BC; therefore the figure described on BC is equal to the similar and similarly described figures on BA, AC (V. B).

PROPOSITION XXXII. THEOREM.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

Let ABC, DCE, be two triangles which have the two sides BA, AC, proportional to the two CD, DE, namely, BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE; BC and CE are in a straight line.



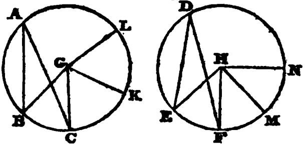
Because AB is parallel to DC, and the straight line AC meets them, the alternate angles BAC, ACD, are equal (I. 29); for the same reason, the angle CDE is equal to the angle ACD; wherefore also BAC is equal to CDE; and because the triangles ABC, DCE, have one angle at A equal to one at D, and the sides about these angles proportionals, namely, BA to AC, as CD to DE, the triangle ABC is equiangular to DCE (VI. 6); therefore the angle ABC is equal to the angle DCE; and the angle BAC was proved to be equal to ACD; therefore the whole angle ACE is equal to the two angles ABC, BAC; add the common angle ACB, then the angles ACE, ACB, are equal to the angles ABC, BAC, ACB; but ABC, BAC, ACB, are equal to two right angles (I. 32); therefore also the angles ACE, ACB, are equal to two right angles; therefore (I. 14) BC and CE are in a straight line.

PROPOSITION XXXIII. THEOREM.

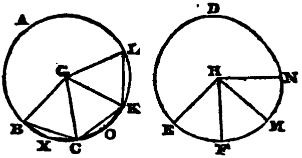
In equal circles, angles, whether at the centres or circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.

Let ABC, DEF, be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF, at their circumferences; as the arc BC to the arc EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and also the sector BGC to the sector EHF.

Take any number of arcs CK, KL, each equal to BC, and any number whatever FM, MN, each equal to EF; and join GK, GL, HM, HN. Because the arcs BC, CK, KL, are all equal, the angles BGC, CGK, KGL, are also all equal (III. 27); therefore what multiple soever the arc BL is of the arc BC, the same multiple is the angle BGL of the angle BGC; for the same reason, whatever multiple the arc EN is of the arc EF, the same multiple is the angle EHN of the angle EHF; and if the arc BL be equal to the arc EN, the angle BGL is also equal to the angle EHN; and if the arc BL be greater than EN, likewise the angle BGL is greater than EHN; and if less, less. There being then four magnitudes, the two arcs BC, EF, and the two angles BGC, EHF, and of the arc BC, and of the angle BGC, have been taken any equimultiples whatever, namely, the arc BL, and the angle BGL; and of the arc EF, and of the angle EHF, any equimultiples whatever, namely, the arc EN, and the angle EHN; and it has been proved, that, if the arc BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if less, less; as therefore the arc BC to the arc EF, so is the angle BGC to the angle EHF (V. Def. 10); but as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF (V. 15); for each is double of each (III. 20); therefore, as the arc BC is to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.



Also, as the arc BC to EF, so is the sector BGC to the sector EHF. Join BC, CK, and in the arcs BC, CK, take any points X, O, and join BX, XC, CO, OK; then, because in the triangles GBC, GCK, the two sides BG, GC, are equal to the two CG, GK, and that they contain equal angles; the base BC is equal to the base CK (I. 4), and the triangle GBC to the triangle GCK; and because the arc BC is equal to the arc CK, the remaining part of the whole circumference of the circle ABC, is equal to the remaining part of the whole circumference of the same circle; wherefore the angle BXC is equal to the angle COK; and the segment BXC is therefore similar to the segment COK (III. Def. 8); and they are upon equal straight lines BC, CK; but similar segments of circles upon equal straight lines are equal to one another (III. 24); therefore the segment BXC is equal to the segment COK; and the triangle BGC is equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK; for the same reason, the sector KGL is equal to each of the sectors BGC, CGK; in the same manner, the sectors EHF, FHM, MHN, may be proved equal to one another; therefore, what multiple soever the arc BL is of the arc BC, the same multiple is the sector BGL of the sector BGC; for the same reason, whatever multiple the arc EN is of EF, the same multiple is the sector EHN of the sector EHF; and if the arc BL be equal to EN, the sector BGL is equal to the sector EHN; and if the arc BL be greater than EN, the sector BGL is greater than the sector EHN; and if less, less; since, then, there are four magnitudes, the two arcs BC, EF, and the two sectors BGC, EHF, and of the arc BC, and sector BGC, the arc BL and sector BGL are any equal multiples whatever; and of the arc EF, and sector EHF, the arc EN and sector EHN are any equimultiples whatever; and that it has been proved, if the arc BL be greater than EN, the sector BGL is greater than the sector EHN; and if equal, equal; and if less, less; therefore (V. Def. 10), as the arc BC is to the arc EF, so is the sector BGC to the sector EHF.



Cor. 1.—Similar sectors of the same, or of equal circles, are equal.

Cor. 2.—An angle at the centre of a circle, is to four right angles, as the arc on which it stands, to the circumference of the circle.

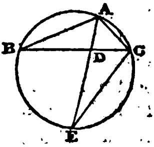
For an angle at the centre, is to one right angle, as the are subtending the former one to a quadrantal arc; therefore (V. 4) the angle at the centre, is to four right angles, as the arc subtending it, to the whole circumference.

PROPOSITION B. THEOREM.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; the rectangle BA · AC is equal to the rectangle BD · DC, together with the square of AD.

Describe the circle ACB about the triangle (IV. 5), and produce AD to the circumference in E, and join EC; then because the angle BAD is equal to the angle CAE, and the angle ABD to the angle AEC (III. 21), in the same segment; the triangles ABD, AEC, are equiangular to one another; therefore as BA to AD, so is EA to AC (VI. 4), and consequently the rectangle BA · AC is equal to the rectangle EA · AD (VI. 16); that is, to the rectangle ED · DA, together with the square of AD (II. 3); but the rectangle ED · DA is equal to the rectangle BD · DC (III. 35); therefore the rectangle BA · AC is equal to the rectangle BD · DC, together with the square of AD.

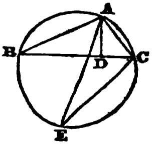


PROPOSITION C. THEOREM.

If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC; the rectangle BA · AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.

Describe (IV. 5) the circle ACB about the triangle, and draw its diameter AE, and join EC; because the right angle BDA is equal to the angle ECA in a semicircle (III. 3), and the angle ABD to the angle AEC in the same segment (III. 21); the triangles ABD, AEC, are equiangular; therefore, as BA to AD, so is EA to AC (VI. 4); and consequently the rectangle BA · AC is equal to the rectangle EA · AD (VI. 16).

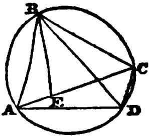


PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD; the rectangle contained by AC, BD, is equal to the two rectangles contained by AB, CD, and by AD, BC.

Make the angle ABE equal to the angle DBC; *add to* each of these the common angle EBD, then the *angle ABD* is equal to the angle EBC; and the angle BDA is equal to the angle BCE (III. 21), because they are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE; wherefore (VI. 4) as BC is to CE, so is BD to DA; and consequently the rectangle BC · AD is equal to the rectangle BD · CE (VI. 16); again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC (III. 21), the triangle ABE is equiangular to the triangle BCD; as therefore BA to AE, so is BD to DC; wherefore the rectangle BA · DC is equal to the rectangle BD, AE; but the rectangle BC · AD has been shown equal to the rectangle BD · CE; therefore the whole rectangle AC · BD is equal to the rectangle AB · DC, together with the rectangle AD · BC (II. 1.)

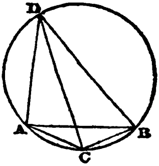


PROPOSITION E. THEOREM.

If a segment of a circle be bisected, and from the extremities of the base of the segment, and from the point of bisection straight lines be drawn to any point in the circum-

ference, the sum of the two lines drawn from the extremities of the base, will have to the line drawn from the point of bisection, the same ratio which the base of the segment has to the base of half the segment.

Let ABD be a circle, of which AB is a segment bisected in C, and from A, C, and B to D, any point whatever in the circumference, let AD, CD, BD, be drawn; the sum of the two lines AD and DB has to DC the same ratio that BA has to AC.



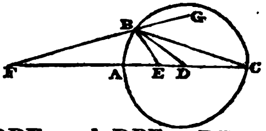
For since ACBD is a quadrilateral inscribed in a circle, of which the diagonals are AB and CD, the rectangles AD · CB and DB · AC are together equal to the rectangle AB · CD (VI. D). But the rectangle AD · BC is equal to the rectangle AD · AC, because BC is equal to AC; and therefore the two rectangles AD · AC and BD · AC are equal to the rectangle AB · CD. But the two rectangles AD · AC and BD · AC are the rectangle contained by AC, and the sum of the lines AD and DB (II. 1); wherefore the rectangle contained by AC and the sum of the lines AD, DB, is equal to the rectangle AB · CD; and because the sides of equal rectangles are reciprocally proportional (VI. 14), the sum of AD and DB is to DC as AB to AC.

PROPOSITION F. THEOREM.

If two points be taken in the diameter of a circle, such that the rectangle contained by the segments intercepted between them and the centre of the circle be equal to the square of the semidiameter; and if from these points two straight lines be inflected to any point whatsoever in the circumference of the circle, the ratio of the lines inflected will be the same with the ratio of the segments intercepted between the two first-mentioned points and the circumference of the circle.

Let ABC be a circle, of which the centre is D, and in DA produced, let the points E and F be such that the rectangle ED · DF is equal to the square of AD; from E and F to any point B in the circumference, let EB, FB, be drawn; FB : BE : : FA : AE.

Join BD, and because the rectangle FD · DE is equal to the square of AD, that is, of DB, FD : DB : : DB : *DE* (VI. 17). The two triangles FDB, BDE, have therefore the sides proportional that are about the common angle D; therefore they are equiangular (VI. 6), the angle DEB being equal to the angle DBF, and DBE to DFB. Now, since the sides about these equal angles are also proportional (VI. 4), FB : BD : : BE : ED, and alternately (V. 16), FB : BE : : BD : ED, or FB : BE : : AD : DE. But because FD : DA : : DA : DE, by division (V. 17), FA : DA : : AE : ED, and alternately, FA : AE : : DA : ED. Now, it has been shown that FB : BE : : AD : DE; therefore (V. 11) FB : BE : : FA : AE.

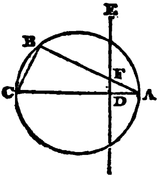


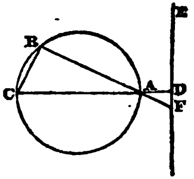
Cor.—If AB be drawn, because FB : BE : : FA : AE, the angle FBE is bisected by AB (VI. 3). Also, since FD : DC : : DC : DE, by composition (V. 18), FC : DC : : CE : ED; and since it has been *shown that* FA : AD (DC) : : AE : ED, therefore, by *equality*, FA : AE : : FC : CE. But FB : BE : : FA : AE; therefore, FB : BE : : FC : CE; so that if FB be produced to G, and if BC be drawn, the angle EBG is bisected by the line BC (VI. A).

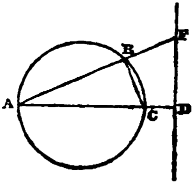
PROPOSITION G. THEOREM.

If, from one extremity of the diameter of a circle, a chord be drawn, and a perpendicular be drawn to the diameter, so as to cut it and the chord either internally or externally, the rectangle under the diameter and its segment reckoned from that extremity, is equal to the rectangle under the chord and its corresponding segment.

Let ABC be a circle, of which AC is a diameter, let DE







be perpendicular to the diameter AC, and let AB meet DE in F; the rectangle BA · AF is equal to the rectangle CA · AD.

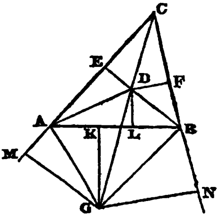
Join BC, and because ABC is an angle in a semicircle, it is a right angle (III. 31). Now, the angle ADF is also a right angle (Hyp.); and the angle BAC is either the same with DAF, or vertical to it; therefore the triangles ABC, ADF, are equiangular, and BA : AC : : AD : AF (VI. 4); therefore also the rectangle BA · AF, contained by the extremes, is equal to the rectangle AC · AD, contained by the means (VI. 16).

PROPOSITION H. THEOREM.

The area of a triangle is a mean proportional between the rectangle under the semiperimeter and its excess above the base, and that under its excesses above the other two sides.

Let ABC be any triangle, its area is a mean proportional between the rectangles $S(S-AB)$ and $(S-AC)(S-BC)$, where S is the semiperimeter.

Bisect the angles at its base, and those adjacent to them by the lines AD, BD, AG, and BG; and join their intersections D and G with C; draw from D and G the perpendiculars DE, DF, DL, and GM, GK, GN, upon the sides and sides produced.



In the triangles ADE, ADL, the angles at A are equal, and the angles at E and L are also equal, being right angles, and AD is common to them, therefore (I. 26) $AL = AE$ and $DL = DE$. It is proved in a similar manner that $BL = BF$, and $DL = DF$; also that $AM = AK$, $GM = GK$; and $BN = BK$, and $GN = GK$; and since DE and DF are each equal to DL, therefore $DE = DF$; and similarly $GM = GN$. Again (I. 47), $CE^2 + ED^2 = CD^2 = CF^2 + FD^2$, and $ED^2 = FD^2$, for $ED = FD$, therefore $CE^2 = CF^2$, and $CE = CF$; and hence (I. 8) the triangles CED, CFD, are equal in every respect; and therefore the angle at C is bisected by CD. The triangles CMG, CNG, are similarly proved (I. 47 and I. 8) to be equal in every respect; hence the angle at C is bisected by CG; therefore the lines CG, CD, coincide.

Since $AK = AM$ and $BK = BN$, therefore $CM + CN$ = perimeter, and $CM = CN = S$, the semiperimeter; and $AM = S - AC$. Also $2CE + 2AE + 2AM$ = perimeter $= 2CF + 2AL + 2BL$, and $CE = CF$, $AE = AL$, therefore $2AM = 2BL$, and therefore AM or $AK = BL$, and therefore $AL = BK$; and because $AM = BL$, therefore $ME = AB$, and $CE = S - AB$; and because $AE = AL = BK = BN$, therefore $AE = S - BC$.

Again, since ED is parallel to MG (I. 29), $CE : ED = CM : MG$; but $CM : CM = ED : ED$, therefore $CE \cdot CM : ED \cdot CM = CM \cdot ED : MG \cdot ED$ (VI. 23, Cor. 1). But the triangle ABC is equal to the three ADC, ADB, CDB, which are respectively equal to half the rectangles $AC \cdot DE$, $AB \cdot DL$, $BC \cdot DF$ (I. 41), the sum of which is $= ED \cdot CM$. And since angles $EAL + MAK$ = two right angles, $EAD + MAG$ = one right angle $= AGM + MAG$ (I. 32), and therefore $EAD = AGM$, and the angles at E and M are right angles, hence the triangles AGM, EAD, are similar, and $AE : ED = MG : AM$; therefore $AE \cdot AM = MG \cdot ED$. The former analogy therefore becomes $CM \cdot CE : ABC = ABC : AM \cdot AE$ or $S\,(S - AB) : ABC = ABC : (S - AC)\,(S - BC)$.

Cor.—If the sides opposite to the angles A, B, C, be respectively denoted by a, b, and c, then

$$s\,(s - c) : ABC = ABC : (s - a)\,(s - b).$$

Schol.—This proposition affords the useful rule, by which the area of a triangle may be computed, when its three sides are given.

EXERCISES.

. Lines that meet any three parallel lines, are cut proportionally.

2. If a straight line be divided into three segments such, that the rectangle under the whole line and the middle segment, is equal to that under the extreme segments, the line is cut harmonically.

3. If from the extremities of the base of a triangle, lines

be drawn bisecting the opposite sides, they will divide each other in the ratio of two to one.

4. If a line be drawn parallel to the base of a triangle to meet the sides, and the alternate extremities of this line and of the base be joined, the line drawn from the vertex through the intersection of the connecting lines, will bisect the base, and will be cut harmonically.

5. The inclination of two chords of a circle is measured by half the sum or half the difference of the intercepted arcs, according as they intersect internally or externally.

This property of the circle suggested a considerable improvement in the form of astronomical angular instruments.

6. If three lines be in continued proportion, the first is to the third, as the square of the difference between the first and second, to the square of the difference between the second and third.

7. If a line bisect the angle adjacent to the vertical angle of a triangle, and meet the base produced, the difference between the square of that line and the rectangle under the external segments of the base, is equal to the rectangle under the sides of the triangle.

This may be considered as another case of VI. B.

8. If two tangents and a secant be drawn to a circle from a point without it, and the points of contact be joined by a straight line, the secant will be cut harmonically.

9. If two triangles have two angles together equal to two right angles, and other two angles equal, the sides about their remaining angles are proportional.

10. If a line be drawn through any point in the base of an isosceles triangle, so as to cut off from one side and add to the other, equal segments, it will be bisected by the base.

11. If from the extremities of the base of a triangle, lines be drawn through any point in the perpendicular to meet the sides, lines joining the points of section of the sides with that of the base, will make equal angles with the base.

12. If from the angular points of a triangle, lines be drawn through any point within it to meet the opposite sides, and if from the point of section of the base, lines be drawn through the other two points of section to meet a

line drawn through the vertex parallel to the base, the intercepted portion of the latter is bisected in the vertex.

13. Harmonicals cut all straight lines that intersect them harmonically.

14. To cut a given line harmonically.

15. If lines be drawn from the angular points of a triangle, through any point within it, to meet the opposite sides, and the points of section be joined, the former lines will be cut harmonically.

16. If through any point within or without a triangle, lines be drawn from the angular points to meet the opposite sides, and if lines joining the points of section be produced to meet the sides produced if necessary, the three points of concourse are in one line.

17. If a semicircular and quadrantal arc have a common chord, the lune contained by these arcs is equal to an isosceles right-angled triangle, having the chord for its hypotenuse.

This is the lune of Hippocrates.

18. Given one side of a triangle, the angle *opposite to it*, and the sum of the other two sides, to construct the triangle.

19. Given the altitude of a triangle, the difference of the angles at the base, and the sum or difference of the sides, to construct it.

20. The altitude, the difference of the segments of the base, and either the sum or difference of the two sides of a triangle, are given to construct it.

21. Given the altitude, the vertical angle, and the perimeter of a triangle, to construct it.

22. Given the base, the altitude, and the sum or difference of the sides of a triangle, to construct it.

23. If arcs of different circles have a common chord, lines diverging from one of its extremities will cut the arcs proportionally.

24. If on two opposite sides of a rectangle semicircles be described towards the same parts, the mixtilineal space contained by their circumferences and the other two sides of the rectangle, is equal to the rectangle.

ON THE QUADRATURE OF THE CIRCLE, AND THE RECTIFICATION OF ITS CIRCUMFERENCE.

DEFINITIONS.

1. The determination of a square equal to a given surface, is called its *quadrature*.

2. The determination of a straight line equal to a curve line, is called its *rectification*.

3. A *mixed line* is composed of straight and curve lines.

4. A *mixtilineal* space is a space contained by a mixed line.

5. The *inclination* of a straight line and a curve is the angle contained by the former, and a tangent to the latter at the point of intersection.

6. The *supplemental* chord of an arc is the chord of its defect from a semicircumference.

AXIOMS.

1. Of all lines that can join two points, there must be at least one such that no other is less than it; and if there be only one such, it is the least.

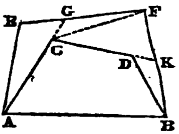
2. If any number of lines fulfil certain conditions, and if a line fulfilling the same conditions can always be found less than any of them, except one, this one must be the least.

PROPOSITION I. THEOREM.

If two rectilineal figures be on the same side of the same base, and if one of them be wholly encompassed by the other, and be also concave internally, the sum of its sides is less than that of the other.

Let the figures ACDB, AEFB, be upon the same base AB, then $AE + EF + FB > AC + CD + DB$.

For, produce AC to G; join C, F, and produce CD to K; then (I. 20) $AE + EG > AG$, $CG + GF > CF$, $CF + FK > CK$, and $DK + KB > DB$. Therefore $AE + EF + FB > AG + GF + FB$, which is $> AC + CF + FB$, which is $> AC + CK + KB$, which is $> AC + CD + DB$, or $AE + EF + FB > AC + CD + DB$.



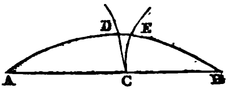
And, in a similar manner, the proposition is proved for polygons.

PROPOSITION II. THEOREM.

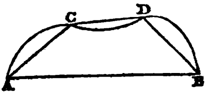
Of all the lines, straight and curve, that can join two points, the straight line is the least.

Let A, B, be any two points, the straight line AB is the shortest line that can join them.

1. Let the curve line ADB, concave towards AB, join them. Then take any point C in AB, and from the centres A and B, with the radii AC, CB, respectively, describe the arcs CD, CE, passing through C, and cutting ADB in D and E. Then, since the arcs touch at C (III. 12), and cannot touch at any other point (III. 13), the point E must be without the arc CD, and D without the arc CE; and the line DE between them must be of some length. Now, if the points D and E be made to coincide with C, A and B remaining fixed, the curve lines AD, EB, would connect A and B; and hence a line shorter than ADEB can connect these points. The same can be proved of any other concave line, therefore the straight line AB is shorter than any of them (Ax. 1).



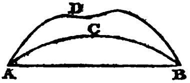
2. Let a line ACDB, partly concave and partly convex towards AB, connect the points. Let AC, DB, be concave, and CD the convex portion. Draw the straight lines AC, CD, DB. Then (by 1st case) the straight lines AC, CD, DB, are less than the curve lines AC, CD, DB, respectively; and the sum of the former therefore less than the whole curve line; hence a line shorter than it has been found.



3. Let the crooked line ACDB join A and B. Then, if AD be joined by a straight line, $AC + CD > AD$, and $AD + DB > AB$; therefore the sum of the straight lines AC, CD, and DB, are greater than AB.

Since, therefore, a line shorter than any of the curve or crooked lines that join A and B can be found, but none shorter than the straight line can be found, therefore it is the shortest (Ax. 1).

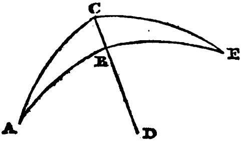
Cor. 1.—If two points A, B, be joined by a straight line AB, and a curve line ACB, or a mixed line, which is concave towards AB; ACB is the least of all the lines that lie on the same side of AB, that join A and B, and that do not lie between ACB and AB.



For if ADB be any other line, then by joining two points in it by a straight line, a shorter line will be found than it, joining A and B, and fulfilling the other conditions; and this will be the case whether ADB be composed of straight lines, or be a mixed line, or a curve line. And as this can be proved of every line fulfilling the required conditions, except of ACB; therefore (Ax. 2) ACB is the least.

Cor. 2.—If two intersecting curves, whose curvatures lie in one and the same direction, be cut by a straight line, inclined to the interior at an angle not greater than a right angle, the arc which it intercepts on this line from the point of intersection, will be less than the corresponding arc of the other line.

Let AB, AC, intersect in A, and be cut by CD, so that the angle at B is less than a right angle, then $AC > AB$.



For, if CE, BE, be curves equal to AC, AB, in every respect, on the other side of CD, but in a reverse position, and having respectively the same inclinations to CD; then, since the angles at B do not exceed two right angles, ABE is concave towards D, and hence (last Cor.) $ACE > ABE$, and therefore $AC > AB$.

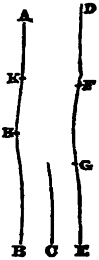
Cor. 3.—Hence the circumference of a circle is greater than the perimeter of any inscribed polygon, and less than that of any one circumscribing it.

PROPOSITION III. THEOREM.

If from the greater of two unequal magnitudes there be taken away its half, and from the remainder its half, and so on, there shall at length remain a magnitude less than the least of the proposed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken away its half, and from the remainder its half, and so on, there shall at length remain a magnitude less than C.

For C may be multiplied so as at length to become greater than AB. Let it be so multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH equal to its half, and from the remainder AH take HK equal to its half, and so on, until there be as many divisions in AB as there are in DE; and let the divisions in AB be AK, KH, HB; and the divisions in ED be DF, FG, GE; and because DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is equal to its half; therefore the remainder GD is greater than the remainder HA. Again, because GD is greater than HA, and that GF is not greater than the half of GD, but HK is equal to the half of HA; therefore the remainder FD is greater than the remainder AK; and FD is equal to C; therefore C is greater than AK; that is, AK is less than C.

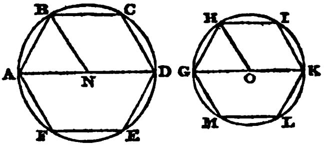


PROPOSITION IV. THEOREM.

Equilateral polygons of the same number of sides inscribed in circles, are similar, and are to one another as the squares of the diameters of the circles.

Let ABCDEF and GHIKLM be two equilateral polygons of the same number of sides inscribed in the circles ABD and GHK; ABCDEF and GHIKLM are similar, and are to one another as the squares of the diameters of the circles ABD, GHK.

Find N and O the centres of the circles; join AN and BN, as also GO and HO, and produce AN and GO till they meet the circumferences in D and K.



Because the chords AB, BC, CD, DE, EF, FA, are all equal, the arcs AB, BC, CD, DE, EF, FA, are also equal (III. 28); for the same reason, the arcs GH, HI, IK, KL, LM, MG, are all equal, and they are equal in number to the others; therefore, whatever part the arc AB is of the whole circumference ABD, the same is the arc GH of the circumference GHK; but the angle ANB is the same part of four right angles that the arc AB is of the circumference ABD (VI. 33, Cor. 2); and the angle GOH is the same part of four right angles that the arc GH is of the circumference GHK; therefore the angles ANB, GOH, are each of them the same part of four right angles, and therefore they are equal to one another. The isosceles triangles ANB, GOH, are therefore equiangular (VI. 6), and the angle ABN equal to the angle GHO; in the same manner, by joining NC, OI, it may be proved that the angles NBC, OHI, are equal to one another, and to the angle ABN; therefore the whole angle ABC is equal to the whole GHK; and the same may be proved of the angles BCD, HIK, and of the rest; therefore the polygons ABCDEF and GHIKLM are equiangular to one another; and since they are equilateral, the sides about the equal angles are proportionals; the polygon ABCDEF is therefore similar to the polygon GHIKLM; and because similar polygons are as the squares of their homologous sides (VI. 20), the polygon ABCDEF is to the polygon GHIKLM as the square of AB to the square of GH; but because the triangles ANB, GOH, are equiangular, the square of AB is to the square of GH, as the square of AN to the square of GO (VI. 4, and 22, Cor.), or as four times the square of AN to four times the square of GO, that is, as the square of AD to the square of GK; therefore, also, the polygon ABCDEF is to the polygon GHIKLM as the square of AD to the square of GK; and they have also been shown to be similar.

Cor.—Every equilateral polygon inscribed in a circle is also equiangular. For the isosceles triangles, which have their common vertex in the centre, are all equal and similar; therefore the angles at their bases are all equal, and the angles of the polygon are therefore also equal.

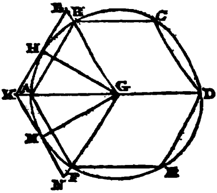
PROPOSITION V. PROBLEM.

The side of any equilateral polygon inscribed in a circle being given, to find the side of a polygon of the same number of sides described about the circle.

Let ABCDEF be an equilateral polygon inscribed in the circle ABD; it is required to find the side of an equilateral polygon of the same number of sides described about the circle.

Find G the centre of the circle; join GA, GB; bisect the arc AB in H; and through H draw KHL touching the circle in H, and meeting GA and GB produced in K and L; KL is the side of the polygon required.

Produce GF to N, so that GN may be equal to GL; join KN, and from G draw GM at right angles to KN; join also HG.



Because the circumference AB is bisected in H, the angle AGH is equal to the angle BGH; and because KL touches the circle in H, the angles LHG, KHG, are right angles; therefore there are two angles of the triangle HGK, equal to two angles of the triangle HGL, each to each; but the side GH is common to these triangles; therefore they are equal, and GL is equal to GK. Again, in the triangles KGL, KGN, because GN is equal to GL, and GK common, and also the angle LGK equal to the angle KGN; therefore the base KL is equal to the base KN; but because the triangle KGN is isosceles, the angle GKN is equal to the angle GNK, and the angles GMK, GMN, are both right angles by construction; wherefore the triangles GMK, GMN, have two angles of the one equal to two angles of the other, and they have also the side GM common; therefore they are equal, and the side KM is equal to the side MN, so that KN is bisected in M; but KN is equal to KL, and therefore their halves KM and KH are also equal; wherefore, in the triangles GKH, GKM, the two sides GK and KH are equal to the two GK and KM, each to each; and the angles GKH, GKM, are also equal, therefore GM is equal to GH; wherefore the point M is in the circumference of the circle; and because KMG is a right angle, KM touches the circle; and in the same manner, by joining the centre and the other angular points of the inscribed polygon, an equilateral polygon may be described about the circle, the sides of which will each be equal to KL, and will be equal in number to the sides of the inscribed polygon; therefore KL is the side of an equilateral polygon described about the circle of the same number of sides with the inscribed polygon ABCDEF; which was to be found.

Cor.—Because GL, GK, GN, and the other straight lines drawn from the centre G to the angular points of the polygon described about the circle ABD, are all equal; if a circle be described from the centre G, with the distance GK, the polygon will be inscribed in that circle; and therefore it is similar to the polygon ABCDEF (4).

PROPOSITION VI. THEOREM.

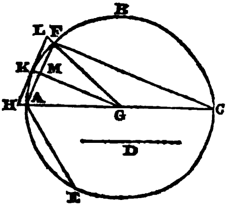
A circle being given, two similar polygons may be found, the one described about the circle, and the other inscribed in it, which shall differ from one another by a space less than any given space.

Let ABC be the given circle, and the square of D any given space; a polygon may be inscribed in the circle ABC, and a similar polygon described about it, so that the difference between them shall be less than the square of D.

In the circle ABC apply the straight line AE equal to D, and let AB be a fourth part of the circumference of the circle. From the circumference AB take away its half, and from the remainder its half, and so on till the circumference AF is found less than the circumference AE. Find the centre G; draw the diameter AC, as also the straight lines

AF and FG; and having bisected the circumference AF in K, join KG, and draw HL touching the circle in K, and meeting GA and GF produced in H and L; join CF.

Because the isosceles triangles HGL and AGF have the common angle AGF, they are equiangular (VI. 6), and the angles GHK, GAF, are therefore equal to one another; but the angles GKH, CFA, are also equal, for they are right angles; therefore the triangles HGK, ACF, are likewise equiangular.

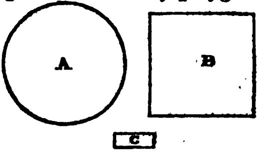


And because the arc AF was found by taking from the arc AB its half, and from that remainder its half, and so on, AF will be contained a certain number of times exactly in the arc AB, and therefore it will also be contained a certain number of times exactly in the whole circumference ABC; and the straight line AF is therefore the side of an equilateral polygon inscribed in the circle ABC; wherefore, also, HL is the side of an equilateral polygon of the same number of sides described about ABC (5). Let the polygon described about the circle be called M, and the polygon inscribed be called N; then, because these polygons are similar (5, Cor.), they are as the squares of the homologous sides HL and AF (VI. 20, Cor. 3), that is, because the triangles HLG, AFG, are similar, as the square of HG to the square of AG, that is, of GK; but the triangles HGK, ACF, have been proved to be similar, and therefore the square of AC is to the square of CF, as the polygon M to the polygon N; and, by conversion, the square of AC is to its excess above the square of CF; that is, to the square of AF, as the polygon M to its excess above the polygon N; but the square of AC, that is, the square described about the circle ABC, is greater than the equilateral polygon of eight sides described about the circle, because it contains that polygon; and, for the same reason, the polygon of eight sides is greater than the polygon of sixteen, and so on; therefore the square of AC is greater than any polygon described about the circle by the continual bisection of the arc AB;

it is therefore greater than the polygon M. Now, it has been demonstrated, that the square of AC is to the square of AF, as the polygon M to the difference of the polygons; therefore, since the square of AC is greater than M, the square of AF is greater than the difference of the polygons. The difference of the polygons is therefore less than the square of AF; but AF is less than D; therefore the difference of the polygons is less than the square of D, that is, than the given space.

Cor. 1.—Because the polygons M and N differ from one another more than either of them differs from the circle, the difference between each of them and the circle is less than the given space, namely, the square of D; and, therefore, however small any given space may be, a polygon may be inscribed in the circle, and another described about it, each of which shall differ from the circle by a space less than the given space.

Cor. 2.—The space B, which is greater than any polygon that can be inscribed in the circle A, and less than any polygon that can be described about it, is equal to the circle A. If not, let them be unequal; and first, let B exceed A by the space C; then, because the polygons described about the circle A are all greater than B, by hypothesis, and because B is greater than A by the space C, therefore no polygon can be described about the circle A, but what must exceed it by a space greater than C, which is absurd. In the same manner, if B be less than A by the space C, it is shown that no polygon can be inscribed in the circle A, but is less than A by a space greater than C, which is also absurd; therefore A and B are not unequal; that is, they are equal to one another.

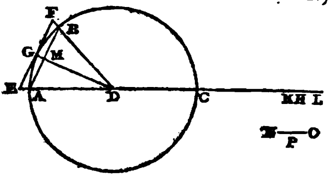


PROPOSITION VII. THEOREM.

Every circle is equal to the rectangle contained by the semidiameter, and a straight line equal to half the circumference.

Let ABC be a circle, of which the centre is D, and the diameter AC; if in AC produced there be taken AH *equal to half the circumference*, the circle is equal to the rectangle contained by DA and AH.

Let AB be the side of any equilateral polygon inscribed in the circle ABC; bisect the circumference *AB in* G, and through G draw EGF touching the circle, and meeting DA produced in E, and DB produced in F; EF will be the side of an equilateral polygon described about the circle ABC (5). In AC produced take AK equal to half the perimeter of the polygon whose side is AB, and AL equal *to* half the perimeter of the polygon whose side is EF; then AK will be less, and AL greater than the straight line AH (2, Cor. 3). Now, because in the triangle EDF, DG is drawn perpendicular to the base, the triangle *EDF is* equal to the rectangle contained by DG and the *half of EF;* and as the same is true of all the other equal triangles having their vertices in D, which make up the polygon described about the circle; therefore the whole polygon is equal to the rectangle contained by DG and AL, half the perimeter of the polygon, or by DA and *AL; but* AL is greater than AH, therefore the rectangle DA · AL is greater than the rectangle DA · AH; the rectangle DA · AH is therefore less than the rectangle DA · AL, that is, than any polygon described about the circle ABC.



Again, the triangle ADB is equal to the rectangle contained by DM the perpendicular and one-half of the base AB, and it is therefore less than the rectangle contained by DG, or DA, and the half of AB; and as the same is true of all the other triangles having their vertices in D, which make up the inscribed polygon, therefore the whole of the inscribed polygon is less than the rectangle contained by DA, and AK half the perimeter of the polygon. Now, the rectangle DA · AK is less than DA · AH; therefore the polygon whose side is AB is still less than DA · AH; and the rectangle DA · AH is therefore greater than any polygon inscribed in the circle ABC; but the same rectangle DA · AH has been proved to be less than any polygon described about the circle ABC; therefore the rectangle DA · AH is equal to the circle ABC (6, Cor. 2). Now, DA is the semidiameter of the circle ABC, and AH the half of its circumference.

Cor. 1.—Hence, a polygon may be described about a circle, the perimeter of which shall exceed the circumference of the circle by a line that is less than any given line. Let NO be the given line. Take in NO the part NP less than its half, and less also than AD, and let a polygon be described about the circle ABC, so that its excess above ABC may be less than the square of NP (6, Cor. 1). Let the side of this polygon be EF; and since, as has been proved, the circle is equal to the rectangle DA · AH, and the polygon to the rectangle DA · AL, the excess of the polygon above the circle is equal to the rectangle DA · HL, therefore the rectangle DA · HL is less than the square of NP; and therefore, since DA is greater than NP, HL is less than NP, and twice HL less than twice NP, wherefore twice HL is still less than NO; but HL is the difference between half the perimeter of the polygon whose side is EF, and half the circumference of the circle; therefore twice HL is the difference between the whole perimeter of the polygon and the whole circumference of the circle. The difference, therefore, between the perimeter of the polygon and the circumference of the circle, is less than the given line NO.

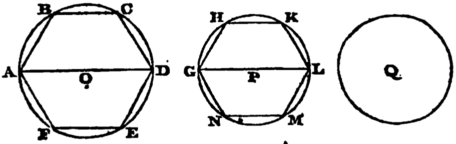
Cor. 2.—Hence, also, a polygon may be inscribed in a circle, such that the excess of the circumference above the perimeter of the polygon may be less than any given line. This is proved like the preceding.

Cor. 3.—Hence (VI. 33), the area of a sector is equal to half the rectangle under its arc and the radius.

PROPOSITION VIII. THEOREM.

Circles are to one another in the duplicate ratio, or as the squares of their diameters.

Let ABD and GHL be two circles, of which the diameters are AD and GL; the circle ABD is to the circle GHL, as the square of AD to the square of GL.



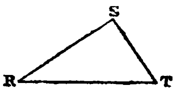
Let ABCDEF and GHKLMN be two equilateral polygons of the same number of sides inscribed in the circles ABD, GHL; and let Q be such a space that the square of AD is to the square of GL, as the circle ABD to the space Q. Because the polygons ABCDEF and GHKLMN are equilateral, and of the same number of sides, they are similar (4), and are as the squares of the diameters of the circles in which they are inscribed; therefore, as the square of AD to the square of GL, so is the polygon ABCDEF to the polygon GHKLMN; but as the square of AD to the square of GL, so is the circle ABD to the space Q; therefore the polygon ABCDEF is to the polygon GHKLMN, as the circle ABD *to the space* Q; but the polygon ABCDEF is less than the *circle ABD*, therefore GHKLMN is less than the space Q. (V. 14); wherefore the space Q is greater than any polygon inscribed in the circle GHL.

In the same manner, it is demonstrated that Q is less than any polygon described about the circle GHL; wherefore the space Q is equal to the circle GHL (4, Cor. 2). Now, by hypothesis, the circle ABD is to the space Q, as the square of AD to the square of GL; therefore *the circle* ABD is to the circle GHL, as the square of AD to the square of GL.

Cor. 1.—Hence, the circumferences of circles are to one another as their diameters.

Let the straight line X be equal to half the circumference of the circle ABD, and the straight line Y to half the circumference of the circle GHL; and because the rectangles AO · X and CP · Y are equal to the circles ABD and GHL (7); therefore the rectangle AO · X is to the rectangle GP · Y, as the square of AD to the square of GL, or as the square of AO to the square of GP; therefore, alternately the rectangle AO · X is to the square of AO, as the rectangle GP · Y to the square of GP; but rectangles that have equal altitudes are as their bases, therefore X is to AO as Y to GP; and again, alternately, X is to Y as AO to GP, and taking the doubles of each, the circumference ABD is to the circumference GHL as the diameter AD to the diameter GL.

Cor. 2.—The circle that is described upon the side of a right-angled triangle opposite to the right angle, is equal to the two circles described on the other two sides; for the circle described upon SR is to the circle described upon RT as the square of SR to the square of RT; and the circle described upon TS is to the circle described upon RT as the square of ST to the square of RT; wherefore the circles described on SR and ST are to the circle described on RT as the squares of SR and ST to the square of RT (V. 24); but the squares of RS and ST are equal to the square of RT; therefore the circles described on RS and ST are equal to the circle described on RT.

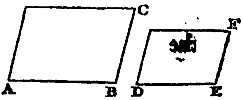


PROPOSITION IX. THEOREM.

Equiangular parallelograms are to one another as the products of the numbers proportional to their sides.

Let AC and DF be two equiangular parallelograms, and let M, N, P, and Q, be four numbers, such that AB : BC : : M : N; AB : DE : : M : P, and AB : EF : : M : Q, and therefore, by equality, BC : EF : : N : Q. The parallelogram AC is to the parallelogram DF as MN to PQ.

Let NP be the product of N into P, and the ratio of MN to PQ will be compounded of the ratios of MN to NP, and of NP to PQ; but the ratio of MN to NP is the same with that of M to P, because MN and NP are equimultiples of M and P; and for the same reason, the ratio of NP to PQ is the same with that of N to Q; therefore the ratio of MN to PQ is compounded of the ratios of M to P, and of N to Q. Now, the ratio of M to P is the same with that of the side AB to the



side DE; and the ratio of N to Q the same with that of the side BC to the side EF; therefore the ratio of MN to PQ is compounded of the ratios of AB to DE, and of BC to EF; and the ratio of the parallelogram AC to the parallelogram DF is compounded of the same ratios; therefore the parallelogram AC is to the parallelogram DF as *MN*, *the* product of the numbers M and N, to PQ the product of the numbers P and Q.

Cor. 1.—Hence, if GH be to KL as the number M to number N; the square described on GH will be to the square described on KL as MM the square of the number *M to* NN the square of the number N.

Cor. 2.—If the square on GH be to the square on KL as a number to a number, the line GH will be to the line KL as the square root of the first number to the square root of the second number; for, if GH were to KL in any other ratio, the square described on GH would not be to the square described on KL *in the* ratio supposed.

Schol.—This proposition is demonstrated here only in the case of commensurable lines, as the fifteenth proposition of the fifth book, on which part of the demonstration depends, is proved only in the case of integral multiples. Before it be proved for incommensurable lines, it must first be proved that M : P = MN : NP when N is an interminate number (Ad. V. Def. 3, and Al. Pr. VIII.*) The numbers proportional to the sides are the numbers denoting the number of times that the unit of measure is contained in these sides (Ad. V. Def. 5).

PROPOSITION X. THEOREM.

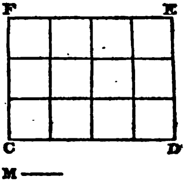
The area of a rectangle is equal to the square of the unit of measure multiplied by the product of the numbers denoting the number of times that this unit is contained in two of its adjacent sides.

Let CDEF be a rectangle, M the unit of measure, and let its sides CD, DE, be denoted by A and B, and the number of times that M is contained in A and B respectively by a and b, then $A \cdot B = abM^2$.

---

* The reference here is to the additional fifth book and the algebraical principles at the end of the volume.

1. When M is contained a certain number of times in A and B, as 4 and 3 times respectively, it is obvious that CE may be divided into a number of squares each $= M^2$, by drawing lines parallel to the sides through the points of division, that there will be as many rows as there are units in b, and as many squares in each row as there are units in a, therefore generally $A \cdot B = abM^2$.*



2. When a and b are terminate, but not integral numbers, let $a = \frac{m}{n}$ and $b = \frac{p}{q}$, these are equal to $\frac{mq}{nq}$, and $\frac{np}{nq}$. Take a line M′ such that $M = nqM'$, then A and B will contain M′ as often as there are units in mq and np. Call these products a' and b' respectively, then (by 1st case) $A \cdot B = a'b'M'^2 = mqnpM'^2$. But $M^2 = nq \cdot nq \cdot M'^2$ (also by 1st case) $= n^2q^2M'^2$, or $M'^2 = \frac{M^2}{n^2q^2}$; therefore $A \cdot B = \frac{mnpqM^2}{n^2q^2} = \frac{mp}{nq}M^2. = \frac{m}{n} \cdot \frac{p}{q}M^2.$

3. Let b be an interminate number. If $A \cdot B$ is not $= abM^2$, let $A \cdot B = ab'M^2$; and let b' be interminate and $> b$, and let b'' be an intermediate terminate number. Then (by 2d case) $A \cdot B'' = ab''M^2$, where B″ is such a line that $B : B'' = b : b''$; and since $b < b''$, therefore $B < B''$ (V. 14), and $A \cdot B < A \cdot B''$, and therefore $ab'M^2 < ab''M^2$. But $b' > b''$, therefore $ab'M^2$ is also $> ab''M^2$, which is impossible; therefore b' is not $> b$; and it may be similarly shown that it is not less; therefore $A \cdot B = abM^2$.

4. Let a and b be both interminate. Then if $A \cdot B$ is not $= abM^2$, let $A \cdot B = ab'M^2$, b' being $> b$ as in the 3d case, and let B″ and b'' be taken as in that case, then $A \cdot B''$

---

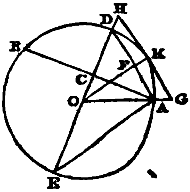
* Before the student can understand any of the following cases, it will be necessary for him to be acquainted with the algebraical principles at the end.

$= ab''M^2$ (by 3d case), but $A \cdot B'' > A \cdot B$, therefore $ab''M^2 > ab'M^2$; and since $b' > b''$, therefore $ab''M^2 < ab'M^2$, which is impossible. Hence b' is not $> b$, and similarly it may be shown not to be less; therefore $A \cdot B = abM^2$.

PROPOSITION XI. THEOREM.

The rectangle under the radius of a circle, and the sum of the diameter and supplemental chord of an arc, is equal to the square of the supplemental chord of half that arc.

Let ABE be a circle, AB any arc, AD its half; AD, AB, their chords, and OF, OC, perpendiculars upon them from the centre; then $2EC \cdot EO = AE^2$.



For, produce EC to D, and since the angle EAD is a right angle (III. 31), and therefore (I. 29) EA is parallel to OF; therefore (VI. 4) the triangles OFD, EAD, are similar, and therefore $DE : DO = EA : OF$. But $DE = 2DO$, therefore $EA = 2OF$, or the perpendicular on AD from the centre, is half of the supplemental chord EA. Similarly it is shown that OC is half the supplemental chord of the arc AB. The triangles CEA and EAD are similar, the angles at A and C being right angles, and that at E common; therefore $CE : EA = EA : ED$, and therefore (VI. 16) $CE \cdot ED = EA^2$, but $2CE = 2EO + 2OC = DE + 2OC$, and $2OC$ is the spplemental chord of AB; and $CE \cdot ED = 2CE \cdot EO$; therefore $2CE \cdot EO = AE^2$.

Cor. 1.—If P be the perpendicular on the chord of an arc, and Q that on the chord of half that arc, and the radius $= 1$; then $4Q^2 = 1\ (2 + 2P)$, or $2Q^2 = \frac{1}{2}(2 + 2P) = 1 + P$.

For $AE = 2OF = 2Q$, and $2CE = 2EO + 2OC = 2 + 2P$, and $EO = 1$.

Cor. 2.—Let C denote the chord on which P is the perpendicular, then $\frac{1}{4}C^2 = 1 - P^2$.

For if $AD = C$, $AD^2 = 4AF^2$, and $AF^2 = OA^2 - OF^2$.

Cor. 3.—Let $R =$ the side of an inscribed regular polygon, and S that of the corresponding circumscribed one, and P the perpendicular on the former; then $S = \frac{R}{P}$.

For, if AD, GH, be the sides of the polygons, then $OF : OK = AD : GH$, or $P : 1 = R : S$ (Ad. V. 7), and $S : 1 = R : P$ (Ad. V. 10), or $S = \frac{R}{P}$.

COR. 4.—The triangles DCA, DAE, are similar, and therefore $ED : DA = DA : DC$, or $AD^2 = ED \cdot DC = ED\,(OD - OC) = 2\,(1 - OC)$.

Schol.—In these four corollaries and the next proposition, the radius is supposed $= 1$, and numbers proportional to the other lines are taken instead of the lines. When the square of a number is found, the number itself can of course be found by extracting the square root of the former.

PROPOSITION XII. PROBLEM.

To find the approximate ratio of the diameter of a circle to its circumference as nearly as may be required.

This may be done by calculating first the apothem of an inscribed regular polygon, by taking some polygon, the length of the side of which is accurately known, as that of a square or hexagon. Then calculate (11, Cor. 1) the apothems of the polygons found by doubling successively the number of sides, till at last the apothem of a polygon of a sufficient number of sides be found; its side may then be found (11, Cor. 2); and then the side of the corresponding circumscribed polygon may be found (11, Cor. 3). The perimeters of the two last polygons will be found by multiplying one of their sides by the number of sides, and it will be found that the numbers expressing their values are the same for a certain number of places of figures; and since the value of the circumference of the circle is intermediate between these, it will be accurately expressed to that number of places by the figures common to both these former numbers; and this approximate value may, by the same method, be carried to any degree of accuracy required.

If a hexagon be taken for the first polygon, its side $R = 1$, therefore the square of its apothem is $= 1 - \frac{1}{4}R^2 = 1 - \frac{1}{4} = \frac{3}{4}$; and if its apothem be called A, $A^2 = \frac{3}{4}$. If B be the apothem on the polygon of 12 sides, then (11, Cor. 1) $2B^2 = 1 + A$, and as A is found, B^2 and therefore B will be found. If C be the apothem on the inscribed polygon of 24 sides, then, similarly, $2C^2 = 1 + B$, and if B be known, C^2 and therefore C can be found. In the same manner, the values of D, E, F, &c., the apothems of inscribed polygons of 48, 96, 192, &c. sides, may be found. The following are the values of the apothems of eight successive polygons, beginning with the hexagon.

No. of sides of polygon.		Apothem.	Value of apothem.
6	$A^2 = 1 - \frac{1}{4} = \frac{3}{4}$ and	A	$= \cdot 86602540$
12	$2B^2 = 1 + A$	B	$= \cdot 96592583$
24	$2C^2 = 1 + B$	C	$= \cdot 99144486$
48	$2D^2 = 1 + C$	D	$= \cdot 99785892$
96	$2E^2 = 1 + D$	E	$= \cdot 99946458$
192	$2F^2 = 1 + E$	F	$= \cdot 99986614$
384	$2G^2 = 1 + F$	G	$= \cdot 99996653$
768	$2H^2 = 1 + G$	H	$= \cdot 99999163$
1536	$2K^2 = 1 + H$	K	$= \cdot 99999791$

Now (11, Cor. 4) if AB be the side of the polygon of 768 sides, then $OC = H$, and $AD^2 = 2(1 - H)$, and $AD = \cdot 004090612$, which is the side of the inscribed polygon of 1536 sides. But (11, Cor. 3) if $AD = R$ and $GH = S$, then $S = \frac{R}{K} = \cdot 004090618 =$ the side of the corresponding circumscribing polygon. And the perimeters of these two polygons, found by multiplying these two sides by 1536, are, respectively, 6·283180032 and 6·283189248; and consequently the approximate value of the circumference of the circle, carried to 6 places in the decimal part, is 6·283185.

If the first inscribed regular polygon be a square, instead of a hexagon, its apothem is the square root of $\frac{1}{2}$ or of ·5, and the apothems and sides of the successive inscribed and circumscribed polygons of 8, 16, 32, &c. sides may be calculated in the same manner as above. If the sides of these polygons be found, and then their areas be calculated by multiplying their perimeter by half their apothem, these areas carried to 7 places in the decimal, will be found to be the same as in the subjoined table.

Number of sides.	Area of inscribed polygon.	Area of circumscribed polygon.
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

Since the areas of the last inscribed and circumscribed polygons agree as far as the seventh decimal place, this must be also the area of the circle, which is always of an intermediate value; and since the area of a circle is equal to the rectangle under its circumference and half its radius, therefore the radius being 1, the value of the semicircumference is 3·1415926, which is also the ratio of the circumference of any circle to its diameter, carried to the seventh decimal place.

The value of this ratio may be carried in the same manner to any required degree of approximation; but much more expeditious methods of effecting this are afforded by analytical principles. This approximate value was found by Archimedes to be $3\frac{1}{7}$ or 22 to 7; by Peter Metius 355 to 113; it was carried by Vieta to 11 figures; by Adrianus Romanus to 17; by Ludolph Van Ceulen to 36; by Abraham Sharp to 74; by Machin to 100; and by De Lagny to 127 figures. This ratio, carried to 36 figures, is 3·14159,26535,89793,23846,26433,83279,50288.

OF GEOMETRICAL MAXIMA AND MINIMA.

DEFINITIONS.

1. Of all magnitudes that fulfil the same conditions, the greatest is called a *maximum*, and the least a *minimum*.

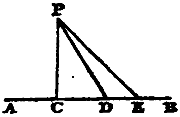
2. Figures that have equal perimeters are said to be *isoperimetrical*.

AXIOM.

Of all isoperimetrical figures, there must be at least one such that its area is not exceeded by that of any other, and if there be only one such, it is the maximum.

PROPOSITION I. THEOREM.

Of all straight lines that can be drawn from a given *point* to a given straight line, the perpendicular is the *least*.



Let P be the given point, and AB the line; draw PC perpendicular to AB, and any other line PD meeting AB, $PC < PD$.

For, since the angles at C are right angles, angle PDC is less than a right angle (I. 17); therefore $PC < PD$.

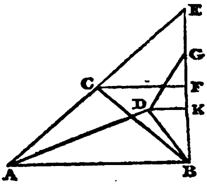
COR.—A line nearer to the perpendicular is less than one more remote.

For angle PDE is greater than a right angle (I. 16), and PEC is less (I. 17), therefore $PD < PE$.

PROPOSITION II. THEOREM.

Of all triangles having the same base and equal perimeters, that is a maximum, whose undetermined sides are equal.

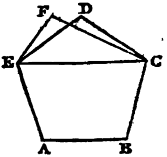
Let ABE, ADB, be two triangles having the same base AB, and $AC + CB = AD + DB$, then if $AC = CB$, the triangle $ABC > ADB$.



For, draw BE perpendicular to AB, and draw CF, DK, perpendicular to BE, make FE = FB and KG = KB, join CE and DG. Then the triangles CFE, CFB, have BF = FE and FC common, and the angles at F right angles, therefore CE = CB. For a similar reason, DG = DB. Therefore AC + CE = AC + CB, and AD + DG = AD + DB. Now angle ECF = BCF = CBA (I. 29) = CAB (I. 5), therefore ECF + ACF = CAB + ACF = two right angles (I. 29), therefore AC, CE, are in the same straight line. But AD + DG $>$ AG, if AG were joined, therefore AG $<$ AE, and hence the point G is nearer to the perpendicular AB than E is (1, Cor.), hence GB $<$ EB, and therefore KB $<$ FB. Therefore the altitude of the triangle ADB being less than that of ACB, ADB $<$ ACB (VI. 1, Cor. 2.)

PROPOSITION III. THEOREM.

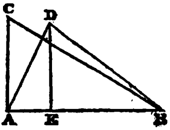
Of all isoperimetrical polygons of the same number of sides, the equilateral polygon is a maximum.



Let ABCDE be the maximum polygon, then if two of its sides as ED, DC, be not equal, let EF, FC, be drawn equal. Then the triangle EFC $>$ EDC (2); and therefore the given polygon is less than ABCFE, which is contrary to hypothesis. Therefore ED = DC; and it may be similarly proved that any two adjacent sides are equal.

PROPOSITION IV. THEOREM.

Of all triangles having only two sides given, that is the greatest in which these sides are perpendicular.

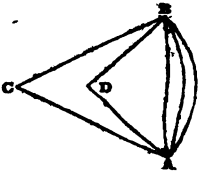


Let ABC, ABD, be two triangles having the common base AB and AC = AD, then if AC be perpendicular to AB, ABC $>$ ABD.

For, draw DE perpendicular to AB, then (1) DE $<$ AD, and therefore DE $<$ AC; hence the triangle ACD $>$ ADB (VI. 1, Cor. 2.)

PROPOSITION V. THEOREM.

Of all the angles subtended at the centres of different circles by equal chords, that in the least circle is the greatest.

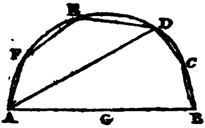


Let the same chord AB subtend the angles D, C, at the centres of two circles, of which CB, DB, are the radii, then $D > C$ (I. 21.)

PROPOSITION VI. THEOREM.

There is only one way of forming a polygon, all of whose sides are given, except one, and inscribed in a semicircle, of which the unknown side is a diameter.

Let the sides of the polygon ABCDEF be all given except AB, and inscribed in a semicircle AEB, of which AB is the diameter. If, now, a greater circle were taken, the angles at the centre subtended by the chords BC, CD, DE, EF, FA, would be respectively less than the angles subtended by them at G, the centre of AEB, that is, they would be less than two right angles; and therefore the extremities A, B, of the given sides would not fall at the extremities of a diameter. If a less circle were taken, the sum of the angles would exceed two right angles, and the same consequence would follow. Hence the polygon in question can be inscribed in only one semicircle.

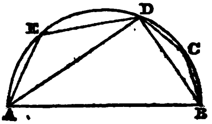


Schol.—The order of the sides AF, FE, &c. may be altered in any manner, the diameter AB and the area of the polygon remaining the same. For the segments cut off by these sides are always the same, and the area of the polygon is equal to that of the semicircle diminished by these segments.

PROPOSITION VII. THEOREM.

Among all polygons whose sides are all given but one, that is the maximum whose sides can be inscribed in a semicircle, of which the unknown side is the diameter.

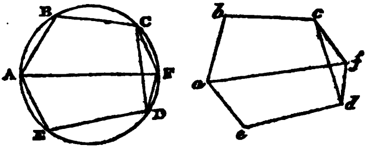
Let ABCDE be the greatest polygon, which can be formed with the given sides AE, ED, DC, CB, and the last AB assumed of any magnitude. Join AD, DB; then if the angle ADB were not a right angle, by making it so the triangle ADB would be increased (4), and the parts AED, DCB, remaining the same, the whole polygon would be increased. But the given polygon, being a maximum, cannot be augmented; therefore the angle ADB must be a right angle. In the same manner, the angles AEB, ACB, must be right angles, and the polygon is inscribed in a semicircle.



PROPOSITION VIII. THEOREM.

Of all polygons formed with given sides, that which can be inscribed in a circle is a maximum.

Let ABCDE, *abcde*, be two polygons, whose corresponding sides are equal, namely, AB $=$ *ab*, BC $=$ *bc*, &c. and let the former be inscribed in a circle, but the latter incapable of being so inscribed, the former is the greater.



Draw from A the diameter AF; join CF, FD, and make the triangle *cfd* $=$ CFD, in every respect, so that *cf* $=$ CF, *df* $=$ DF, and join *af*.

The polygon ABCF $>$ *abcf* (7), unless the latter can be inscribed in a semicircle having *af* for its diameter, in which case the two polygons would be equal. For a similar reason, the polygon AEDF $>$ *aedf*, unless in the case of a similar exception, by which the polygons would be equal. Hence, the whole polygon ABCFDE $>$ *abcfde*, unless the latter can be inscribed in a circle. But it cannot; therefore the former is the greater. Take from both the equal triangles CFD, *cfd*, and there remains the polygon ABCDE $>$ *abcde*.

Schol.—It may be shown, as in the sixth proposition, that there is only one circle in which the polygon can be inscribed, and therefore only one maximum polygon; and

this polygon will have the same area in whatever order the sides be arranged.

PROPOSITION IX. THEOREM.

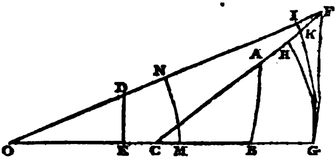
Of all isoperimetrical polygons, having the same number of sides, the regular polygon is the greatest.

For (3) the maximum polygon is equilateral, and (8) it can be inscribed in a circle, it is therefore a regular polygon.

PROPOSITION X. THEOREM.

Of two isoperimetrical regular polygons, the one having the greater number of sides is the greater.

Let DE, AB, be half-sides of these polygons, OE, CB, their apothems, lying in the same straight line, and ACB, DOE, the half-angles at their centres. Since these angles are unequal, the sides OD, CA, if produced, will meet in some point F. Draw FG perpendicular to CG, and from the centre O describe the arcs GI, MN, with the radius OG and the radius OM $=$ CG; and describe the arc GH from the centre C with the radius CG.



The polygons being regular, their perimeters are the same multiples of their half-sides, that four right angles is of their half-angles at their centres, but their perimeters are equal; therefore, by direct equality, angle $O : C = DE : AB$. But $O : C = MN : GH$ (VI. 33), therefore $DE : AB = MN : GH$; but $OG : OM$ or $CG = GI : MN$ (Qu.* VIII. Cor. 1), therefore (VI. 23, Cor. 1) $DE \cdot OG : AB \cdot CG = MN \cdot GI : MN \cdot GH = GI : GH$ (VI. 1) if MN be the altitudes of these two rectangles. But the triangles OED, OGF, are similar, therefore $OE : OG = DE : FG$ (VI. 4), therefore $OE \cdot FG = DE \cdot OG$ (VI. 16). Also, from the similar triangles, ABC, FGC, $CB : AB = CG : FG$; and hence $CB \cdot FG = AB \cdot CG$. Therefore $OE \cdot FG : CB \cdot FG = GI : GH$, or (VI. 1) $OE : CB = GI : GH$. But (Qu. II. Cor. 2) $GK > GH$, therefore GI is still $> GH$, therefore $OE > CB$. But the perimeters of the polygons being equal, their areas will be proportional to their apothems; therefore the polygon, of which OE is the apothem, is the greater. But its half-angle O is less than that of the other, or it has a greater number of sides; therefore the polygon having the greater number of sides is the greater.

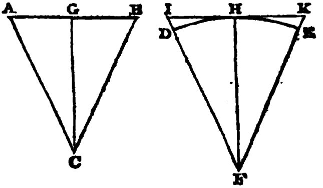
---

* *Qu.* refers to the preceding book on the Quadrature of the Circle.

PROPOSITION XI. THEOREM.

The circle is greater than any polygon of the same perimeter.

Let AG be a half-side, and C the centre of a regular polygon of the same perimeter as the circle, of which F is the centre, and DE an arc subtending an angle DFE equal to the angle ACB subtended by the side AB; then the triangle ACB and the sector DFE are equal parts of the whole polygon and circle, and AB and DE are equal parts of their perimeters, and are therefore equal; therefore, if the polygon and circle be respectively denoted by P and C, $P:C=$ triangle ACB : sector $DFE=AG\cdot GC:DH\cdot HF=GC:HF$. But if IHK be a side of the corresponding circumscribed polygon, then by the similar triangles ABC, IFK, $GC:HF=AB$ or $DE:IK$; and therefore $P:C=DE:IK$. But (Qu. II. Cor. 2) $DH<IH$, and therefore $IK>DE$, hence $C>P$. Now P is a regular polygon, and of all isoperimetrical polygons having the same number of sides, the regular polygon is the greatest; but the circle has been proved to be greater than this polygon; it is therefore greater than any other isoperimetrical polygon.



PROPOSITION XII. THEOREM.

Of all polygons having the same area and the same number of sides, the regular polygon has its perimeter a minimum.

Let A and P be the area and perimeter of a regular polygon B, and N the number of its sides, and A', P', N', those of an irregular polygon B', if $A=A'$ and $N=N'$, then $P<P$.

For, let B'' be a polygon similar to B' and A'', P'' and N'' its area, perimeter, and the number of its sides. If $P''=P$ and $N''=N$, then $A'' \angle A$ (9); and therefore if the parts of B'' be proportionally increased till $A''=A$, then its perimeter $=P'$ (VI. 20). But $P' > P''$, and therefore $P' > P$.

PROPOSITION XIII. THEOREM.

Of regular polygons having the same area, that which has the greatest number of sides has the least perimeter.

Let B be a regular polygon, A, P, N, its area, perimeter, and the number of its sides; A', P', N', those of another regular polygon B'; and A'', P'', N'', those of another B'' similar to B'. If $N > N'$ then P is $\angle P'$. For since $N'' \angle N$, if $P''=P$, $A'' \angle A$ (10); and if the parts of B'' be increased proportionally till $A''=A$, then (VI. 20) its perimeter will $=P'$; but $P' > P''$ and $P''=P$, therefore $P' > P$.

Schol.—In accordance with this and other propositions, bees instinctively construct their cells in the form of regular hexagons, which form requires the least quantity of wax.

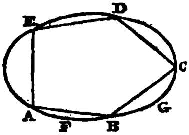
PROPOSITION XIV. THEOREM.

The perimeter of a circle is less than that of any polygon having the same area.

Let A and P be the area and perimeter of a circle B, A' and P' those of a polygon B'; and A'', P'', those of a polygon B'' similar to B'. Then, if $P''=P$, A'' will be $\angle A$ (11); and if the parts of B'' be proportionally increased till $A''=A$, then its perimeter will $=P'$; but $P' > P''$ and $P''=P$, therefore $P' > P$.

PROPOSITION XV. THEOREM.

Of all isoperimetrical plane figures, the circle contains the greatest area.



Let ACE be the maximum figure for a given perimeter. If it is not a circle, let an equilateral polygon be described in it. It is evidently possible for such a polygon to exist in it, so that each of its angular points shall not be in the circumference of the same circle with all the other angular points, and therefore the polygon will be irregular

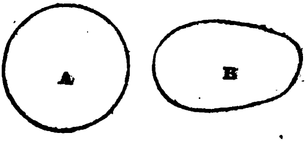
Let a regular polygon P of the same number of sides and the same perimeter be described, then each of its sides will be equal to those of the inscribed polygon. On the sides of P describe segments equal to AFB, BGC, &c., then a curvilinear figure Q will thus be formed, having the same perimeter as the given figure ACE; and since the regular polygon exceeds the other polygon (9), the whole figure Q will exceed the whole ACE. When ACE therefore is not a circle, another figure as Q having the same perimeter, can always be found greater than it; hence the circle is the maximum figure (Ax.)

Schol.—No part of the figure ACE is supposed to be convex internally. Were any portion of it convex, an equal concave arc being substituted for it, the area would thus be increased, while the perimeter is unaltered.

PROPOSITION XVI. THEOREM.

Of all plane figures of the same area, the circle has the least perimeter.

Let the circle A = the figure B, the perimeter of A is less than that of B. For, if not, let another circle C be assumed whose perimeter is equal to that of B; then (15) its area exceeds that of B or of A; and hence its perimeter exceeds that of A (Qu. VIII.)



EXERCISES.

1. A straight line and two points without it being given, to find a point in it such that the sum of the lines drawn from it to the two given points, shall be a minimum, that is, that they shall be less than the sum of any two lines similarly drawn from any other point in it.

2. If any point, except the centre, be taken in the diameter of a circle, of all the chords passing through this point, that is the least which is perpendicular to the diameter.

3. Of all triangles that have the same vertical angle, and whose bases pass through a given point, that whose base is bisected by the point is a minimum.

4. The sum of the four lines drawn to the angular points

of any quadrilateral from the intersection of the diagonals, is less than that of any other four lines similarly drawn from any other point.

5. To find a point in a given line such that the difference of the lines drawn to it from two given points may be a maximum.

6. A straight line and two points on the same side of it being given, to find a point in it such, that the angle contained by lines drawn from it to the given points shall be a maximum.

7. Given three points, to find a fourth such, that the sum of its distances from the given points shall be a minimum.

ON GEOMETRICAL ANALYSIS.

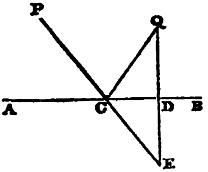
In the method of Geometrical Analysis, the process of demonstration follows an order the reverse of that observed in the ordinary or Synthetic Method. The latter method proceeds from established principles, and, by a chain of reasoning, deduces new principles from these; the former proceeds from the principles that are to be established considered as known, and from these, taken as premises, arrives, by reversing the chain of reasoning, at known principles. The latter method is the didactic method used in communicating instruction; the former is rather employed in the discovery of truth. This method is not exactly that pursued by the mind in analysing any proposition, but it may be considered to be that method reduced to a systematic form.

Magnitudes are said to be *given* when they are actually given or may easily be found; they are said to be *given in position* when their position may be determined; and rectilineal figures are said to be *given in species* when figures similar to them may be found. A circle is said to be given *in position* when its centre is given, and *in magnitude* when its radius is given. A *ratio* is said to be *given* when two quantities having that ratio are given.

The two following propositions are given as examples of this method.

PROPOSITION I. PROBLEM.

Given two points P and Q and a straight line AB, to find a point C in AB such, that lines PC, CQ, drawn to it from P and Q, may make equal angles PCA, QCD, with AB.



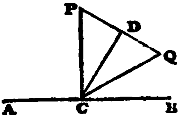
By analysis.—For either of the two given points as Q, draw QD perpendicular to AB, and produce QD to meet PC produced in E; then the angle ECD = ACP (I. 15) = QCD (Hyp.), and the angles CDE, CDQ, are equal, being right angles, and CD is common to the two triangles CDE, CDQ; therefore (I. 26) they are equal in every respect, and hence DE = DQ. But the perpendicular DQ is given, and therefore DE and the point E are given; hence the line PE, and consequently the point C are given.

Schol.—By this analysis, the construction is discovered by which the point C is determined. That C is the required point may now be demonstrated by synthesis or composition, thus:—

By composition.—Draw QD perpendicular to AB, and produce DQ till DE = DQ, and join PE, then C will be the point required.

For, join CQ. Then because QD = DE and DC is common to the two triangles CDE, CDQ, and the angles CDE and CDQ are equal, being right angles, therefore (I. 4) these triangles are every way equal; and hence angle ECD = QCD. But PCA = ECD (I. 15), therefore also PCA = QCD; and C is the point required.

PROPOSITION II. PROBLEM.



A straight line AB, and two points P and Q without it being given, to find a point C in it such, that the two lines CP, CQ, drawn to them from C shall be equal.

Analysis.—Join PQ, and bisect it in D, and join CD. Because CP = CQ and PD = DQ and CD common to the two triangles CDP, CDQ, they are every way equal (I. 8), therefore the angle PDC = QDC. Therefore CD is perpendicular to PQ. But PQ is given, therefore the point D and the perpendicular DC are given, and consequently the point C is given.

It may be easily proved by composition that C is the point required.

Any of the exercises given in the preceding books will serve as exercises in geometrical analysis.

ON PLANE LOCI.

If every point in a straight or curve line, and no other point, fulfil certain conditions, the line is said to be a *locus* of the point.

As a simple illustration of a locus, consider that of a point which is always equally distant from a given point. This is obviously a circle, whose radius is equal to that distance. So the locus of a point, which is always equally distant from a given straight line, is a line parallel to it, and at a distance from it equal to the given distance.

EXERCISES.

1. Find the locus of the vertices of all the triangles that have the same base and one of their sides of a given length.
2. Find the locus of a point that is at equal distances from two given points.
3. Find the locus of a point that is equally distant from two given lines, either parallel or inclined to one another.
4. To find the locus of the vertices of all triangles that have the same base and equal altitudes.
5. To find the locus of the vertices of all triangles that have the same base and one of the angles at the base equal.
6. To find the locus of the angular point opposite to the hypotenuse of all the right-angled triangles that have the same hypotenuse.
7. To find the locus of the vertices of all the triangles that have the same base and equal vertical angles.

8. To find the locus of the vertices of all the triangles that have the same base and equal areas.

9. To find the locus of the vertices of all triangles that have the same base and the sum of the squares of their sides equal to a given square.

10. If straight lines drawn from a given point to a given line be cut in a given ratio, to find the locus of the point of section.

11. To find the locus of the vertices of all the triangles that have the same base and the ratios of their sides equal.

12. If a straight line drawn from a given point, and terminating in the circumference of a given circle, be cut in a given ratio, the locus of the point of section is also the circumference of a given circle.

13. If one of the extremities of straight lines drawn through a given point be terminated in a given straight line, and their other extremities be determined, so that the rectangle under the segments of each line is equal to a given rectangle, the locus of these extremities will be the circumference of a circle; and if one of the extremities be terminated in the circumference of a circle, the locus of the other extremities is a straight line.

ON PORISMS.

A *porism* is an indeterminate problem, or one that admits of an indefinite number of solutions; but every indeterminate problem is not a porism. As a simple example of a porism, let it be required to find a point such, that all straight lines drawn from it to the circumference of a given circle shall be equal. This point is evidently the centre of the circle. Problems of loci and porisms are in many cases mutually convertible. The preceding problem becomes a problem of the latter kind when the centre is given, and it is required to find the locus of all the points that are at a given distance from it.

EXERCISES.

1. Two points being given, to find a third, through which any straight line being drawn, the perpendiculars upon it from the two given points shall be equal.

2. Three points being given, to find a fourth, through which any straight line being drawn, the sum of the perpendiculars upon it from two of the given points on one side of it, shall be equal to the perpendicular on it from the third point.

There are similar porisms when four, five, or any number of points are given, by which another point is to be found such that any straight line being drawn through it, the sum of the perpendiculars from the points on one side of it shall be equal to the sum of those from the points on the other side of it. The point so found is called the point of mean distance; or if the points be considered to be equal physical, or material points, it is called the centre of gravity.

3. A straight line and a circle being given, to find a *point* such that the rectangle under the segments of any *straight* line drawn through it, and limited by these, shall *be* equal to a given rectangle.

This problem is the 13th exercise on loci converted into a porism. Many other problems of loci may be similarly converted; and conversely.

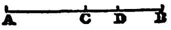
ADDITIONAL SECOND BOOK.

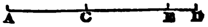
The arrangement of the propositions in this book is nearly the same as in Euclid, and some of them are new. Although the notation be algebraical, no knowledge of algebra is requisite, and the reasoning is strictly geometrical. The enunciations of those propositions that are the same as in Euclid, are differently expressed to accommodate them to the notation. It is of some advantage to vary the enunciations, as the same proposition occurs in geometrical investigations in more than one form, and its identity in different forms is not so easily recognised unless a familiarity be acquired with the various forms of expression. It may be remarked, that when the propositions of the second book are once understood, the figures employed by Euclid, in their demonstrations, readily suggest the proposition; but it will certainly be admitted that the expressions in this book are so obvious by the algebraical notation, that they suggest the propositions, with at least equal facility, and even more so, when the analogy between algebraical products and the expressions for rectangles is considered. The cause of this analogy will be observed in the tenth proposition in the book on the Quadrature of the Circle; and, in fact, it appears by it, that, when the symbols representing the sides of a rectangle denote the numbers proportional to its sides, the algebraical product and the expression for the area of the rectangle are identical.

PROPOSITION I. THEOREM.

A line is equal to the sum or difference of its segments, and twice the mean distance is equal to the difference or sum of the segments, according as the point of section is internal or external.

Let AB be cut equally in C and unequally in D, then $AB = AD \pm DB$ and $2CD = AD \mp DB$, taking the upper sign for the first figure, and the lower for the second.





For (fig. 1) $AB = AD + DB$, and $AD - DB = AC + CD - DB = CB + CD - DB = DB + CD + CD - DB = 2CD$.

And (fig. 2) $AB = AD - DB$, and $AD + DB = AC + CD + DB = CB + CD + DB = 2CD$.

PROPOSITION II. THEOREM.

If, of three lines, the excess of the second above the third, is to be taken away from the first, the remainder will be the same as if the second be taken from the sum of the first and third.

Let A, B, C, represent three lines, then the excess of A above $B - C$ is equal to the excess of $A + C$ above B, or $A - (B - C) = A + C - B$.

For, let $A - (B - C) = D$, and add $(B - C)$ to both these equals, then $A = D + (B - C) = D + B - C$, or $A = D + B - C$. Now, add C to both these equals, and take B from them, and $A + C - B = D$; but $D = A - (B - C)$, therefore $A - (B - C) = A + C - B$.

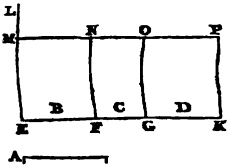
COR.—The proposition applies also to rectilineal figures. For, if they be rectangles of the same altitude, and the lines A, B, C, their bases, they may be added and subtracted in the same manner as the lines; and if they be any other rectilineal figures, they could be reduced to equivalent rectangles of the same altitude (I. 44, 45.)

Schol.—This principle, which is constantly employed *in* algebra, will be found to be of great use in geometry, as it will, in many cases, prevent the necessity of adding or subtracting equal magnitudes, in order to arrive at an obvious result.

PROPOSITION III. THEOREM.

The rectangle under a line, and the sum of any *number* of lines, is equal to the sum of the several *rectangles* contained by that line and each of the other lines.

Let the line $EF = B$, $FG = C$, $GK = D$, and let A be any other line, then $A(B + C + D) = A \cdot B + A \cdot C + A \cdot D$.



Let B, C, D, lie in one straight line; draw EL perpendicular to EK from E, and make $EM = A$; through M draw MP parallel to EK, and through F, G, K, draw FN, GO, KP, each parallel to EM. Then the figures EP, EN, FO, and GP, are rectangles, and $GO = FN = EM = A$, also

$$EP = EN + FO + GP;$$

but $$EP = EM \cdot EK = A(B + C + D),$$

$$EN = ME \cdot EF = A \cdot B,$$

$$FO = NF \cdot FG = A \cdot C,$$

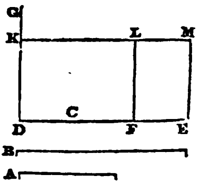
and $$GP = OG \cdot GK = A \cdot D;$$

therefore $A(B + C + D) = A \cdot B + A \cdot C + A \cdot D$.

PROPOSITION IV. THEOREM.

The rectangle under a line, and the difference between two lines, is equal to the difference between the rectangles under that line and each of the other lines.

Let A, B, C, be three lines, and let $B > C$, then $A\,(B - C) = A \cdot B - A \cdot C$.



Let $DE = B$, and $DF = C$, and construct the figure DM, as in the last proposition, making $DK = A$. Then the figures DM, DL, FM, are rectangles, and $LF = KD = A$, and FM is the difference between DM and DL, and $FE = DE - DF = B - C$.

$$\begin{aligned}
&\text{Now,} && FM = DM - DL;\\
&\text{but} && FM = LF \cdot FE = A \cdot (B - C),\\
& && DM = KD \cdot DE = A \cdot B,\\
&\text{and} && DL = KD \cdot DF = A \cdot C;\\
&\text{therefore} && A\,(B - C) = A \cdot B - A \cdot C.
\end{aligned}$$

PROPOSITION V. THEOREM.

The rectangle under the sum of two lines and one of them, is equal to the sum of the rectangle under the two lines and the square of the latter.

Let A and B be two lines, then $(A + B)\,B = A \cdot B + B^2$.

For (3*) $B\,(A + B) = A \cdot B + B \cdot B$, or $(A + B)\,B = A \cdot B + B^2$.

PROPOSITION VI. THEOREM.

The rectangle under the difference between two lines and one of them, is equal to the difference between the rectangle under these lines and the square of the latter.

Let A and B be the two lines, then if $A > B$, $(A - B)\,B = A \cdot B - B^2$.

For (4) $B\,(A - B) = A \cdot B - B \cdot B$, or $(A - B)\,B = A \cdot B - B^2$.

---

* The numbers refer to the propositions in this book.

PROPOSITION VII. THEOREM.

The square of the sum of two lines is equal to the sum of the rectangles under that sum and each of the lines.

Let A, B, be two lines, then $(A+B)^2=(A+B)A+(A+B)B$.

For $(A+B)^2=(A+B)(A+B.)$ Let $A+B=D$; then $(A+B)(A+B)=D(A+B)=D\cdot A+D\cdot B$ (3) $=(A+B)A+(A+B)B$. Therefore $(A+B)^2=(A+B)A+(A+B)B$.

PROPOSITION VIII. THEOREM.

The square of the difference between two lines is *equal* to the difference between the rectangles under that difference and each of the lines.

Let the two lines be A and B, then if $A > B$, $(A-B)^2=(A-B)A-(A-B)B$.

For, let $A-B=D$, then $(A-B)^2=(A-B)(A-B)=D(A-B)=DA-DB$ (4) $=(A-B)A-(A-B)B$.

PROPOSITION IX. THEOREM.

The square of the sum of two lines is equal to the squares of these lines and twice the rectangle contained by them.

Let A and B be two lines, then $(A+B)^2=A^2+B^2+2A\cdot B$.

For $(A+B)^2=(A+B)A+(A+B)B$ (7.) But $(A+B)A=A^2+A\cdot B$ (5), and $(A+B)B=A\cdot B+B^2$. Therefore $(A+B)^2=A^2+B^2+2A\cdot B$.

COR.—The square of a line is equal to four times the square of its half.

For, when $A=B$, $(A+B)^2=(A+A)^2=A^2+A^2+2A\cdot A$, or $(2A)^2=4A^2$.

PROPOSITION X. THEOREM.

The square of the difference of two lines is equal to the squares of these lines diminished by twice their rectangle.

Let A, B, be two lines, then if $A > B$, $(A-B)^2=A^2+B^2-2A\cdot B$.

For, by (8) $(A-B)^2=(A-B)A-(A-B)B$. But $(A-B)A=A^2-AB$ (6), and $(A-B)B=A\cdot B$

$-B^2$. Therefore $(A-B)^2 = A^2 - A\cdot B - (A\cdot B - B^2)$. But, subtracting $A\cdot B - B^2$ is the same as adding B^2 and subtracting $A\cdot B$ (2), therefore $(A-B)^2 = A^2 - A\cdot B - A\cdot B + B^2 = A^2 + B^2 - 2\,A\cdot B$.

PROPOSITION XI. THEOREM.

The rectangle under the sum and difference of two lines is equal to the difference of their squares.

Let A and B be the two lines, of which $A > B$, then $(A+B)\,(A-B) = A^2 - B^2$.

For, let $A+B=S$, then $(A+B)(A-B) = S\cdot A - S\cdot B$ (4.) But $S\cdot A = (A+B)\,A = A^2 + A\cdot B$ (5), and $S\cdot B = (A+B)\,B = A\cdot B + B^2$. Therefore $(A+B)\,(A-B) = A^2 + A\cdot B - A\cdot B - B^2 = A^2 - B^2$.

Schol.—The fifth and sixth propositions of the former second book may be easily derived from this one (see II. 5 and 6, and Schol. to 6.)

PROPOSITION XII. THEOREM.

The sum of the squares of two lines is equal to twice the rectangle under them and the square of their difference.

Let A and B be two lines, then if $A > B$, $A^2 + B^2 = 2\,A\cdot B + (A-B)^2$.

For (10) $(A-B)^2 = A^2 + B^2 - 2A\cdot B$, and adding $2\,A\cdot B$ to both, $2A\cdot B + (A-B)^2 = A^2 + B^2$.

Schol.—This is the same as the seventh proposition of the former second book, or its corollary, if $AB = A$, and $BC = B$, and therefore $AC = A - B$.

PROPOSITION XIII. THEOREM.

The square of the sum of two lines is equal to the square of their difference and four times the rectangle contained by them.

Let A and B be two lines, and $A > B$, then $(A+B)^2 = (A-B)^2 + 4\,A\cdot B$.

For $2\,A\cdot B = 4A\cdot B - 2\,A\cdot B$, and adding $A^2 + B_2$ to both, $A^2 + B^2 + 2\,A\cdot B = 4\,A\cdot B + A^2 + B^2 - 2\,A\cdot B$. Or (9 and 10) $(A+B)^2 = 4\,A\cdot B + (A-B)^2$.

Schol.—This proposition is the same as the eighth of the other second book, if $AB = A$ and $BC = B$.

PROPOSITION XIV. THEOREM.

The square of the sum of two lines, together with the square of their difference, is equal to the sum of twice their squares.

Let A and B be two lines, and $A > B$, then $(A+B)^2 + (A-B)^2 = 2A^2 + 2B^2$.

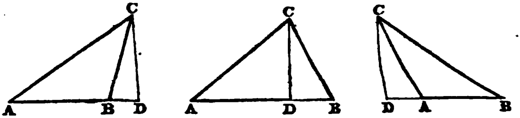
For (9) $(A+B)^2 = A^2 + B^2 + 2A \cdot B$, and (10) $(A-B)^2 = A^2 + B^2 - 2A \cdot B$; therefore $(A+B)^2 + (A-B)^2 = 2A^2 + 2B^2$, for $2A \cdot B - 2A \cdot B = 0$.

Schol.—This proposition is the same as the ninth and tenth of the former second book, if $AC = A$ and $CD = B$, in the former; and $CD = A$, $AC = B$, in the *latter*. It may also be enunciated as in the scholium to the latter.

PROPOSITION XV. THEOREM.

In any triangle, if a perpendicular be let fall from the vertex upon the base, then, according as the angle opposite to one of the sides is obtuse or acute, the square *of that* side is greater or less than the squares of the base *and the* other side, by twice the rectangle contained by the base and the part of it intercepted between that angle and the perpendicular.

Let ABC be any triangle, and CD a perpendicular from C upon AB or AB produced, then $AC^2 = AB^2 + BC^2 \pm 2AB \cdot BD$, taking the upper sign when the angle ABC is obtuse, and the lower when it is acute.



For $AD^2 = AB^2 + BD^2 \pm 2AB \cdot BD$ (9 and 10), add DC^2 to both, then $AD^2 + DC^2 = AB^2 + DB^2 + DC^2 \pm 2AB \cdot BD$. But (I. 47) $AD^2 + DC^2 = AC^2$, and $BD^2 + DC^2 = BC^2$, therefore $AC^2 = AB^2 + BC^2 \pm 2AB \cdot BD$.

Propositions XVI. XVII. XVIII. and XIX. are the same as A, B, C, and D, of the former second book.

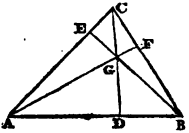
Schol.—Proposition C and Cor. 1, are true also when the perpendicular falls upon the base produced. In Cor. 3, it is obvious, that, if the squares of two lines drawn from two points to the same extremity of the given lines, respectively exceed the squares of the lines drawn from the same points to the other extremity, the two points lie on the same side of the line that bisects the given line perpendicularly; for the greater line is adjacent to the greater segment of the base.

PROPOSITION XX. THEOREM.

If, from the extremities of the base of a triangle, perpendiculars be drawn to the opposite sides, the line joining their intersection and the vertex is perpendicular to the base.

Let ABC be a triangle, and AF, BE, perpendiculars upon the sides, then CGD is perpendicular to the base AB.

For (II. c.) $AB^2 - BC^2 = AG^2 - GC^2$, and $AB^2 - AC^2 = BG^2 - GC^2$; therefore taking equals from equals, $AB^2 - BC^2 - (AB^2 - AC^2) = AG^2 - GC^2 - (BG^2 - GC^2)$, or (2) $AB^2 - BC^2 - AB^2 + AC^2 = AG^2 - GC^2 - BG^2 + GC^2$, or $AC^2 - BC^2 = AG^2 - BG^2$; and therefore CD is perpendicular to AB (II. c. Cor. 3).



ADDITIONAL FIFTH BOOK.

In this additional fifth book a simple and common definition is adopted of a ratio, which is the only intelligible one, and is in fact identical with that of Euclid, although the latter is not expressed in very definite terms; and a simple, obvious, and concise definition of proportion, is also adopted. The critical difficulty of incommensurables is directly encountered in the seventh proposition, and by means of it all the succeeding propositions in proportion may be demonstrated. By this proposition, the doctrine of proportion, in reference to magnitudes, is reduced to numerical proportion; so that whatever proportion exists among magnitudes, will exist among the numbers that are proportional to them; and the proposition being proved in the case of the latter, will also be established in respect to the former. In the theory of arithmetic and algebra, the principles of proportion are established in the case of terminate numbers, and they are assumed to be true in the case of interminate numbers. Instead of this hasty and unsatisfactory induction, it would be more scientific to establish them also in this case. This has been done, as far as is necessary in geometry, in the algebraical principles at the end of this volume. By means of the principle proved in the twenty-seventh proposition of this book, this view of *proportion* can easily be applied to geometry. In several, otherwise excellent treatises on geometry, the proposition that triangles of the same altitude are as their bases, is proved in the case of the bases being incommensurable, by implicitly assuming as an axiom a principle which is identical with the proposition to be proved; and in some other treatises, the application of the principles of proportion are *frequently* encumbered with indirect demonstrations. Euclid's definition of proportion is prolix and remote from our ordinary conception of proportion; but, by an ingenious evasion of the question of incommensurability, he has thus been enabled to give direct, though frequently tedious, demonstrations of the principles of proportion. Were it possible always thus to evade the question, the *method* of Euclid might perhaps be considered to be the best, but *it only* removes the question a stage farther forward, for it is sure to present itself before a complete treatise of geometry can be composed; and, in fact, it occurs in such a form that evasion is no longer possible in proving the proposition (Qu. X.) that the area of a rectangle is equal to the square of the unit of measure, multiplied by the product of the numbers denoting the number of times that this unit is contained in its sides. Since this question, therefore, must be recognised and disposed of in a scientific manner, it would perhaps be the best method to do so at first, since the principles of proportion can by this means be more simply established; as one proof of which, no less than nine propositions, which by the method of Euclid require to be demonstrated, become axioms. The property of numbers, by means of which the propositions in proportion are extended to incommensurables, and by which the algebraical principles, in the case of interminate numbers, are demonstrated, is stated in the scholium after the definitions of this book.

DEFINITIONS.

1. The *ratio* of two quantities is the number of times that the one contains the other.

Thus, if a quantity A contain another B m times, the ratio of A to B is m, and is expressed thus: $A \div B = m$, or $A : B = m$, or $\frac{A}{B} = m$. The lines A and B are not supposed to be represented by numbers.

2. A *proportion* or an *analogy* consists of two equal ratios.

Let $A : B$ be equal to $C : D$, then $A : B = C : D$, is a proportion or an analogy.

3. A *terminate* number is an integer, a mixed number, or a vulgar fraction.

As a mixed number may be reduced to a vulgar fraction, all terminate numbers may be expressed by $\frac{m}{n}$, where m and n are integers; and when the number is an integer, $n = 1$, and the number $= m$.

4. A number that is not terminate, is called an interminate number.

Interminate numbers comprehend interminate decimals, excepting repeaters and circulaters, which can be reduced to vulgar fractions, and are therefore terminate numbers, although, however, in the form of decimal fractions, they are called interminate decimals.

5. A line by which another is measured, that is, which is applied to another line, in order to determine the number of times that the latter contains it, is called a *unit* of *measure*.

6. A compound ratio may be concisely expressed by enclosing the simple component ratios within parentheses, thus,

if the ratio of A to B be compounded of the ratios of C to D, and of E to F, then A : B = (C : D, E : F.)

Schol.—Between two numbers, however near, though both be interminate, an intermediate terminate number may be found. For, if one of the numbers be terminate, then if it be the less, that part of the other, which extends one place farther than the former, is evidently greater; and if the former be the greater, then some portion of the latter, with its last figure increased by unity, will be intermediate. If they be both interminate, then some portion of the less may be found, such, that if its last figure be increased by unity, it will be intermediate. Any vulgar fraction, when converted into a decimal fraction, will be a finite, a repeating, or a circulating decimal, and if the last, the number of circulating decimals will be less than the denominator. The properties of numbers stated in this scholium, so far as I know, have not before been observed.

AXIOMS.

1. Equal magnitudes have the same ratio to *the same* magnitude.

2. Magnitudes that have the same ratio to the same magnitude are equal.

3. The same magnitude has the same ratio to equal magnitudes.

4. Magnitudes, to which the same *magnitude* has the same ratio, are equal.

5. Of two magnitudes, that which has a greater ratio to the same magnitude, is the greater.

6. Of two magnitudes, that to which the same magnitude has a less ratio, is the greater.

7. Of two equal ratios, according as the one is a ratio of equality, majority, or minority, so must the other; or according as the first term of an analogy is equal to, greater or less than, the second, the third is equal to, greater or less than, the fourth.

8. If the first term of an analogy be any multiple or submultiple of the second by a whole number, the third is the same multiple or submultiple of the fourth.

9. **Ratios that are equal to the same ratio are equal to one another.**

The first five propositions in this book are the same as the first, second, third, fifth, and sixth of the former fifth book.

PROPOSITION VI. THEOREM.

If a multiple of a magnitude by any whole number be taken, and then a submultiple of this multiple by any other whole number, the resulting magnitude will be the same, as if the same submultiple had first been taken of the given magnitude, and then the same multiple of this submultiple.

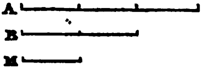
Let A be any magnitude, and m, n, any whole numbers, then a submultiple of mA by n, is equal to a multiple of $\frac{A}{n}$ by m; or $\frac{mA}{n} = m \cdot \frac{A}{n}$.

For, it is evident that the nth part of m times A, is equal to the nth part of A taken m times.

Cor.— $\frac{m}{n} \cdot A$ is either $\frac{mA}{n}$ or $m \cdot \frac{A}{n}$. And similarly, $\frac{m}{n} \cdot nqM = \frac{mnqM}{n}$. If a line A be taken $= nqM$, this becomes evident.

PROPSOITION VII. THEOREM.

The ratio of two lines is the same as that of the numbers, which express the number of times that any third line is contained in them respectively.



Let A and B be any two lines, and M a third, and let M be contained a times in A and b times in B, then $\frac{A}{B} = \frac{a}{b}$, or $A : B = a : b$.

1. Let a and b be integers.

If $a = 3$ and $b = 2$, that is, if A contains M 3 times and B contains it twice, then it is evident that A contains B $1\frac{1}{2}$ times, and that a contains b $1\frac{1}{2}$ times, therefore $A : B = a : b$. The same conclusion is obvious whatever be the values of a and b, provided they be integers; therefore $\frac{a}{b}=\frac{aM}{bM}=\frac{A}{B}$.

If $a=mq$, and $b=np$, m, n, p, and q, being integers, $\frac{a}{b}=\frac{mq}{np}=\frac{mqM}{npM}$.

2. Let $a=\frac{m}{n}$, and $b=\frac{p}{q}$.

Let M′ be a submultiple of M by nq, or $M=nqM'$, then $\frac{a}{b}=\frac{m}{n}\cdot\frac{q}{p}=\frac{mq}{np}=\frac{mqM'}{npM'}$ (1st case). But $mqM'=\frac{mnqM'}{n}$ $=\frac{m}{n}\cdot nqM'$ (6, Cor.*) $=\frac{m}{n}M$. And, similarly, $npM'=$ $\frac{p}{q}M$. Therefore $a:b=\frac{m}{n}M:\frac{p}{q}M$, or $a:b=A:B$.

3. Let a be interminate and b terminate.

If $\frac{A}{B}$ is not $=\frac{a}{b}$, let $\frac{A}{B}=\frac{a'}{b}$. Let $a'>a$, and a'' a terminate number intermediate between a and a'. Then $\frac{a''}{b}=\frac{a''M}{bM}$ (1st or 2d case), and $a''M>aM$ or A; therefore $\frac{a''M}{bM}>\frac{A}{B}$, and hence $\frac{a''}{b}>\frac{a'}{b}$, and therefore $a''>a'$. But a'' is also $<a'$, which is impossible; therefore a' cannot be greater than a, and it may be similarly shown that it cannot be less; therefore $\frac{A}{B}=\frac{a}{b}$.

4. Let a be terminate and b interminate.

This case may be proved similarly to the last.

5. Let a and b be both interminate.

If $\frac{A}{B}$ is not $=\frac{a}{b}$, let $\frac{A}{B}=\frac{a'}{b}$. Let $a'>a$, and a'' a terminate intermediate number. Then $\frac{a''}{b}=\frac{a''M}{bM}$ (4th case.)

---

* The numbers refer to propositions in this book.

The rest of the demonstration is exactly the same as in the 3d. case. Therefore $\frac{A}{B} = \frac{a}{b}$.

PROPOSITION VIII. THEOREM.

The terms of an analogy are proportional by inversion.

Let $A : B = C : D$, then, by inversion, $B : A = D : C$.

For, let a, b, c, and d, be the number of times that a line M is contained in A, B, C, and D, respectively; then $\frac{a}{b} = \frac{c}{d}$; and therefore $\frac{b}{a} = \frac{d}{c}$ (Al. Pr. VII. Cor.), or $b : a = d : c$. Therefore (7) $B : A = D : C$.

PROPOSITION IX. THEOREM.

The terms of an analogy are proportional by alternation.

Let $A : B = C : D$, then, by alternation, $A : C = B : D$.

For (7) $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} \cdot \frac{b}{c} = \frac{c}{d} \cdot \frac{b}{c}$, or (Al. Pr. VIII.) $\frac{a}{c} = \frac{b}{d}$, or $a : c = b : d$. Therefore $A : C = B : D$.

COR.—Also $B : D = A : C$. This is evident (Ax. 9).

PROPOSITION X. THEOREM.

The terms of an analogy are proportional when taken in any regular order.

Let $A : B = C : D$.

1. By alternation, $A : C = B : D$ (9);
2. By inversion, $B : A = D : C$ (8);
3. By the alternation of (2), $B : D = A : C$;
4. By equal ratios, $C : D = A : B$ (Ax. 9);
5. By the alternation of (4), $C : A = D : B$;
6. By the inversion of (4), $D : C = B : A$;
7. By the alternation of (6), $D : B = C : A$.

There are thus eight different analogies formed by transposing the terms in a regular order.

PROPOSITION XI. THEOREM.

If two homologous terms in one analogy be equal to two in another, a new analogy may be formed, by taking the remaining terms in any regular order.

Let A : B = C : D, and E : B = F : D, then A, C, E, and F, will be proportional when taken in any regular order.

For, by alternation (9), A : C = B ; D, and B : D = E : F (9, Cor.), therefore A : C = E : F; and therefore A, C, E, F, taken in any regular order, are proportional (10).

COR.—If there be any number of analogies such that two homologous terms in each is equal to two homologous terms in the following, the remaining terms of the first and last taken in any regular order are proportional.

This may be proved in the same manner as the poposition.

This proposition and corollary are the same as *direct equality* (see V. 22).

PROPOSITION XII. THEOREM.

The terms of an analogy are proportional by composition.

Let A : B = C : D, then, by composition, A + B : B = C + D : D.

For A + B contains B once more than A *does, and* C + D contains D once more than C does, but $\frac{A}{B} = \frac{C}{D}$, therefore A + B contains B as often as C + D contains D, or A + B : B = C + D : D.

PROPOSITION XIII. THEOREM.

The terms of an analogy are proportional by *addition.*

Let A : B = C : D, then, by addition, A : A + B = C : C + D.

For A + B : B = C + D : D, but A : B = C : D, therefore, by direct equality (11), A + B : A = C + D : C, or inversely, A : A + B = C : C + D.

PROPOSITION XIV. THEOREM.

The terms of an analogy are proportional by division.

If A : B = C : D, then, by division, A — B : B = C — D : D.

For A — B contains B just once less than A does, and C — D contains D once less than C does, but $\frac{A}{B} = \frac{C}{D}$, therefore $A - B$ contains B as often as $C - D$ contains D, or $A - B : B = C - D : D$.

Let $B > A$, then inversely, $B : A = D : C$; and by former case, $B - A : A = D - C : C$, therefore, by direct equality, $B - A : B = D - C : D$.

PROPOSITION XV. THEOREM.

The terms of an analogy are proportional by conversion.

Let $A : B = C : D$, then, by conversion, $A : A - B = C : C - D$.

For, by division, $A - B : B = C - D : D$, therefore, by direct equality, $A : A - B = C : C - D$.

PROPOSITION XVI. THEOREM.

If any number of magnitudes be proportional, one antecedent is to its consequent, as the sum of all the antecedents to that of the consequents.

Let $A : B = C : D$, and $C : D = E : F$, then $A : B = A + C + E : B + D + F$.

For $A : A = B : B$, and $C : A = D : B$ (10); and since, from equal ratios, $A : B = E : F$, therefore $E : A = F : B$ (10). Therefore $\frac{A}{A} = \frac{B}{B}$, $\frac{C}{A} = \frac{D}{B}$ and $\frac{E}{A} = \frac{F}{B}$, therefore $\frac{A}{A} + \frac{C}{A} + \frac{E}{A} = \frac{B}{B} + \frac{D}{B} + \frac{F}{B}$, and hence $\frac{A + C + E}{A} = \frac{B + D + F}{B}$, or (7) $A + C + E : A = B + D + F : B$, and (10) $A : B = A + C + E : B + D + F$.

PROPOSITION XVII. THEOREM.

In any analogy, one antecedent is to its consequent, as the difference between the antecedents, to that between the consequents.

If $A : B = C : D$, then $A : B = A - C : B - D$.

For, by alternation, $A : C = B : D$; and, by division, $A - C : C = B - D : D$, and, by direct equality, $A : B = A - C : B - D$.

PROPOSITION XVIII. THEOREM.

The terms of an analogy are proportional by mixing.

This proposition is the same as the twenty-second of the other fifth book.

PROPOSITION XIX. THEOREM.

If the consequents of two analogies be the same, the sum of the first antecedents is to their consequents, as the sum of the other antecedents to their consequents.

This is the same as the twenty-fourth proposition of the other fifth book.

PROPOSITION XX. THEOREM.

Magnitudes have the same ratio as their equimultiples.

Let A and B be two magnitudes, and m any number, then $A : B = mA : mB$.

For $\frac{a}{b} = \frac{ma}{mb}$ (Al. Pr. VIII.), or $a : b = ma : mb$; *therefore* $A : B = mA : mB$.

COR. 1.—Magnitudes are to one another as their equal submultiples. For A and B are equal submultiples of mA and mB, and $mA : mB = A : B$.

COR. 2.—Any equimultiples of two magnitudes have the same ratio as any other equimultiples of the same.

For $A : B = mA : mB$, and $A : B = nA : nB$; *and hence* $mA : mB = nA : nB$.

Schol.—This proposition is proved here when m is any number, integral, fractional, or interminate; but in V. 15, it is proved only when m is an integer.

PROPOSITION XXI. THEOREM.

If any equimultiples be taken of the terms of an analogy, they are proportional.

Let $A : B = C : D$, then $mA : mB = mC : mD$.

For $A : B = mA : mB$ (20), and $C : D = mC : mD$; and therefore $mA : mB = mC : mD$.

PROPOSITION XXII. THEOREM.

If any equimultiples of the terms of one of the ratios of an analogy be taken, and also any equimultiples of the terms of the other ratio, these equimultiples taken in the order of the terms are proportional.

Let $A : B = C : D$, then $mA : mB = nC : nD$.

For $A : B = mA : mB$ (20), and $C : D = nC : nD$; and hence $mA : mB = nC : nD$.

PROPOSITION XXIII. THEOREM.

If any equimultiples be taken of two homologous terms of an analogy, and also any equimultiples of the other two terms, these multiples taken in the order of the terms are proportional.

Let $A : B = C : D$, then $mA : nB = mC : nD$

For, by alternation, $A : C = B : D$, and therefore (22) $mA : mC = nB : nD$, or, by alternation, $mA : nB = mC : nD$.

Schol.—The last three propositions are true whether m or n be terminate or interminate numbers.

PROPOSITION XXIV. THEOREM.

If two terms that are not homologous in one analogy be equal to two that are not homologous in another, a new analogy arises by taking the remaining terms of the first analogy as extremes, and those of the other as means.

Let $A : B = C : D$, and $B : E = F : C$, then $A : E = F : D$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{b}{e} = \frac{f}{c}$, therefore $\frac{a}{b} \cdot \frac{b}{e} = \frac{c}{d} \cdot \frac{f}{c}$ and hence (Al. Pr. IX.) $\frac{ab}{be} = \frac{cf}{dc}$, or $\frac{a}{e} = \frac{f}{d}$ (Al. Pr. VIII.), or $a : e = f : d$; and hence $A : E = F : D$ (7).

COR.—If there be any number of analogies such, that two terms not homologous in each are equal to two terms not homologous in the following, a new analogy will arise by taking the remaining terms of the first as means and those of the last as extremes.

This proposition is commonly called *indirect equality* (see V. 23).

PROPOSITION XXV. THEOREM.

Ratios that are compounded of equal ratios are equal.

This is proved as in proposition F of the other fifth book.

PROPOSITION XXVI. THEOREM.

A compound ratio is equal to the product of its component simple ratios.

If $A:B = \left\{ C:D, E:F, \right\}$ then $\frac{A}{B} = \frac{C}{D} \cdot \frac{E}{F}$.

For, let $C:D = G:H$, and $E:F = H:K$; and since $G:K = \left\{ G:H, H:K, \right\}$ (V. Def. 17) $= \left\{ C:D, E:F, \right\}$ therefore (25) $G:K = A:B$. But $\frac{G}{K} = \frac{g}{k} = \frac{hg}{hk}$ (Al. Pr. VIII.) $= \frac{g}{h} \cdot \frac{h}{k}$ (Al. Pr. IX.) $= \frac{c}{d} \cdot \frac{e}{f}$. And $\frac{A}{B} = \frac{G}{K}$; therefore $\frac{A}{B} = \frac{C}{D} \cdot \frac{E}{F}$.

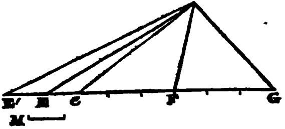
PROPOSITION XXVII. THEOREM.

If four quantities be so related, that when either the first or second increases or diminishes, the third or fourth also respectively increases or diminishes, and also that when the first is a multiple of the second by any terminate number, the third is the same multiple of the fourth; then when the first is a multiple of the second by an interminate number, the third is the same multiple of the fourth.

Let A, B, C, and D, be four quantities such, that when A increases or diminishes so does C, and when B increases or diminishes so does D, and that when A is a multiple of B by any terminate number integral or fractional, C is the same multiple of D; then when A is a multiple of B by an interminate number, C is the same multiple of D; that is, if m be any interminate number, when $A = mB$, $C = mD$.

If not, when $A = mB$, let $C = m'D$. Then if m'' be a terminate number intermediate between m and m', when A becomes $A'' = m''B$, then C becomes $C'' = m''D$. Now, if $m' > m''$, then $m'' > m$; and therefore when the value of A increases from mB to $m''B$, that of C diminishes from $m'D$ to $m''D$, which is contrary to hypothesis. Therefore m' cannot be greater than m, and it may similarly be shown that it cannot be less; therefore m' must $= m$, so that when $A = mB$, $C = mD$.

Schol.—This general principle is of easy and extensive application. As an illustration of its application, let it be required to prove that triangles of the same altitude are as their bases. Let EFH, FHG, be the triangles; and first let their bases be commensurable. Let EF contain the common measure M 4 times, and FG contain it 3 times, therefore EF : FG = 4 : 3 (7). Also, if $Ee = M$, the triangle HEF contains HE*e* 4 times, and FGH contains it 3 times (I. 38); hence EFH : FGH = 4 : 3; and consequently EF : FG = EFH : FGH. Therefore when $EF = \frac{4}{3}$ FG, $EFH = \frac{4}{3}$ FGH; and hence also, when EF and FG are incommensurable, the same proportion exists (27).



It is obvious that the terminate number m'' may always be expressed by some vulgar fraction $\frac{p}{q}$, p and q being integers, and EF or A may be produced to E', so that the measure, which is contained in FG q times, may be contained in E'F p times, and then $A'' = m''B$, and also $C'' = m''D$, for $E'F = A''$, $FG = B$, $E'FH = C''$, and $FGH = D$. This will appear very obvious by giving p and q particular values, as $p = 6$, $q = 5$.

In exactly the same manner it may be proved that angles at the centre, or sectors, of a circle, are proportional to their corresponding arcs. The same principle applies also to similar propositions in solid and spherical geometry.

PLANE TRIGONOMETRY.

Plane Trigonometry treats of those relations that subsist between the sides and angles of triangles, by which their numerical values may be computed.

These relations are established by means of certain lines connected with the angles, called trigonometrical lines. When a sufficient number of the parts of a triangle are known, it may be constructed, and then the unknown parts can be measured. This method, however, which is called construction, or the graphic method, would only give a moderate approximation; but, by trigonometrical computation, the values may be found to any required degree of accuracy. The sides are measured by some line of a determinate length, chosen as the unit of measure, as a foot, a yard, a mile, &c.; and the unit of measure of angles is the 90th part of a right angle, or the 360th part of four right angles. As the angles at the centre of a circle are proportional to the arcs subtending them, these arcs may be taken as their measures. The circumference of a circle is accordingly supposed to be divided into 360 equal parts, called degrees; each of these into 60 equal parts, called minutes; each of these into 60 equal parts, called seconds, and so on. If, therefore, a circle be described from the angular point as a centre, the number of degrees, minutes, &c. contained in the arc intercepted by the lines containing the angle, is the measure of that angle. An angle is also sometimes measured by the length of the intercepted arc of a circle, whose centre is the angular point and radius $= 1$.

DEFINITIONS OF TRIGONOMETRICAL LINES.

1. The *complement* of an arc is its difference from a quadrant; and that of an angle its difference from a right angle.

2. The *supplement* of an arc is its defect from a semicircle; and that of an angle is its defect from two right angles.

3. The *sine* of an arc is a line drawn from one of its extremities, perpendicular to the radius passing through its other extremity.

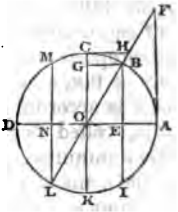
4. The *tangent* of an arc is a line touching it at one extremity, and limited by the radius produced through its other extremity.

5. The *secant* of an arc is that portion of the radius produced, which is intercepted between the extremity of the tangent and the centre.

6. The *versed sine* of an arc is that portion of the radius intercepted between the sine and the extremity of the arc.

7. The *supplemental versed sine*, or *suversed sine*, is the difference between the versed sine and the diameter.

8. The sine, tangent, &c. of the complement of an arc, are concisely termed the *cosine*, *cotangent*, &c. of that arc. These terms, for conciseness, are usually contracted into sin, tan, sec, vers, suvers, cos, cot, cosec, covers, and cosuvers.



Let AB be an arc of a circle, AC a quadrant, O the centre; BE, BG, perpendiculars on the radii OA, OC; AF, CH, tangents at A and C; then BC is the complement of AB; and COB that of angle AOB; DCB the supplement of AB, and angle DOB that of AOB; BE the sine of AB; AF its tangent; OF its secant; AE its versed sine; DE its suversed sine; also BG, CH, OH, the sine, tangent, and secant of BC, or the cosine, cotangent, and cosecant of AB.

COROLLARIES FROM THE DEFINITIONS.

1. The sine of a quadrant, or of a right angle, is equal to the radius.

For CO = sin AC.

2. The tangent of half a right angle is equal to the radius.

For, if angle AOF were half a right angle, so would F (I. 32), and therefore AF would be equal to AO.

3. The sine, tangent, &c. of an arc are equal to those of its supplement.

For BE is the sine of DCB; AF is the tangent of AKI, and therefore of the equal arc DCB; and OF is its secant.

4. The radius is a mean proportional between the tangent and cotangent of an arc.

For FAO, OCH, are similar triangles, and $AF : AO = OC : CH$.

If the arc AB be called A, and the radius OA, R, then $\tan A : R = R : \cot A$, or $R^2 = \tan A \cdot \cot A$.

5. The radius is a mean proportional between the sine and cosecant of an arc.

From similar triangles, $BE : BO = OC : OH$, or $\sin A : R = R : \operatorname{cosec} A$, or $R^2 = \sin A \cdot \operatorname{cosec} A$.

6. The radius is a mean proportional between the cosine and secant of an arc.

By similar triangles, $GB : BO = OA : OF$, or $\cos A : R = R : \sec A$, or $R^2 = \cos A \cdot \sec A$.

7. The tangent is to the radius as the sine to the cosine.

For $FA : AO = BE : EO$, or $\tan A : R = \sin A : \cos A$.

8. The tangent is to the secant as the radius to the cosecant.

For $FA : FO = CO : OH$, or $\tan A : \sec A = R : \operatorname{cosec} A$.

9. The sine of an arc is to the cosine as the secant to the cosecant.

For (Cor. 5 and 6) $\sin A \cdot \operatorname{cosec} A = \cos A \cdot \sec A$, therefore (VI. 16) $\sin A : \cos A = \sec A : \operatorname{cosec} A$.

10. The square of the radius is equal to the squares of the sine and cosine of an arc.

For $OE = GB = \cos A$, and $OB^2 = BE^2 + EO^2 = R^2 = \sin^2 A + \cos^2 A$.

11. The square of the sine of an arc is equal to the difference of the squares of the radius and the cosine.

For (Cor. 10) $\sin^2 A = R^2 - \cos^2 A$.

12. The square of the sine of an arc is equal to the rectangle under the versed sine and suversed sine.

For (III. 35) $BE^2 = AE \cdot ED$, or $\sin^2 A = \operatorname{vers} A \cdot \operatorname{suvers} A$.

13. The square of the secant of an arc is equal to the sum of the squares of the radius and tangent.

For $OF^2 = OA^2 + AF^2$, or $\sec^2 A = R^2 + \tan^2 A$. Similarly $\operatorname{cosec}^2 A = R^2 + \cot^2 A$.

14. The chord of twice an arc is equal to twice the sine of that arc.

For $BI = 2BE$, or chord $2A = 2 \sin A$.

15. The square of the chord of an arc is equal to twice the rectangle under the radius and the versed sine of the arc.

For (Cor. 12) $BE^2 = AE \cdot ED$, therefore $BE^2 + EA^2 = AE \cdot ED + AE^2$, or $AB^2 = AD \cdot AE$, or chord ${}^2A = 2R$. vers A; and if $R = 1$, chord ${}^2A = 2$ vers A.

16. The sine, tangent, and secant of an arc, are the same as those of that arc increased by any number of whole circumferences.

For if to AB any number of whole circumferences be added, the compound arc will terminate at B.

In some of the applications of trigonometry, it is necessary to attend to the signs of the trigonometrical lines. Quantities whose signs are +, are called *positive*, and those whose signs are —, are called *negative*. If a line be measured from a given point or a given line as its origin, it is reckoned *positive* when it lies on one side of its origin, and *negative* when on the opposite side. Thus, if the sines in the first quadrant be measured from the diameter AD, and be reckoned positive, those in the second quadrant, as MN, will also be positive; and those in the third and fourth, as LN, IE, will be negative. So if the cosines, as BG, in the first quadrant, measured from CK be reckoned positive, those in the second and third, being measured from the opposite side of CK, will be negative, and those in the fourth will be positive. In a similar manner, the signs of the tangent, secant, &c. are determined; but this subject belongs properly to Analytical Trigonometry.

If a number as m is to be divided by a number n, the quotient is expressed thus, $\frac{m}{n}$.

If $R = 1$, and the whole circumference $= C$, these corollaries become :—

1. $\sin \frac{1}{4} C = 1$.
2. $\tan \frac{1}{8} C = 1$.
3. $\sin A = \sin (\frac{1}{2} C - A)$.

4. Tan A. cot A = 1, therefore $\tan A = \frac{1}{\cot A}$, and $\cot A = \frac{1}{\tan A}$; or the tangent and cotangent are each others reciprocals.

5. Sin A. cosec A = 1, therefore $\sin A = \frac{1}{\operatorname{cosec} A}$, and $\operatorname{cosec} A = \frac{1}{\sin A}$.

6. Cos A. sec A = 1, therefore $\cos A = \frac{1}{\sec A}$, and $\sec A = \frac{1}{\cos A}$.

7. Tan A. cos A = sin A, or $\tan A = \frac{\sin A}{\cos A}$.

8. Tan A. cosec A = sec A, or $\tan A = \frac{\sec A}{\operatorname{cosec} A}$.

9. $\frac{\text{Sin } A}{\text{Cos } A} = \frac{\sec A}{\operatorname{cosec} A}$.

10. $\text{Sin}^2 A + \cos^2 A = 1$.

11. $\text{Sin}^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$ (II. 5. Cor.); so $\cos^2 A = 1 - \sin^2 A = (1 + \sin A)(1 - \sin A)$.

12. $\text{Sin}^2 A = \text{vers } A . \text{ suvers } A$.

13. $\text{Sec}^2 A = 1 + \tan^2 A$, and $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

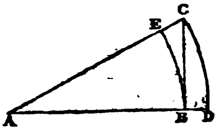
14. Chord 2 A = 2 sin A.

15. $\text{Chord}^2 A = 2 \text{ vers } A$.

16. Sin A = sin (mC + A), where m is any integer.

PROPOSITION I. THEOREM.

If the hypotenuse of a right-angled triangle be made radius, the sides become the sines of the opposite angles, or the cosines of the adjacent angle.



For, let ABC be a right-angled triangle, if its hypotenuse AC be made radius, then BC is the sine of

A; but the angle at C is the complement of A, therefore BC is the cosine of C. By making C the centre, it may be similarly proved that AB is the sine of C, and therefore the cosine of A.

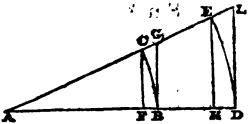
PROPOSITION II. THEOREM.

If one of the sides about the right angle of a right-angled triangle be made radius, the other side becomes the tangent of the opposite angle, and the hypotenuse the secant of the same; or the other side becomes the cotangent of the adjacent angle, and the hypotenuse the cosecant of the same.

When AB (see fig. to Prop. I.) is radius, BC is tangent of A, and AC is secant of A, or BC is cotangent of C, and AC its cosecant.

PROPOSITION III. THEOREM.

The sine, tangent, and other trigonometrical lines of an angle for one radius, are proportional to those for another radius; and the ratios of the corresponding lines are the same as that of the respective radii.



Let A be any angle, and BC, DE, arcs described with the radii AB or AC, and AD or AE; then if CF, EK, are its sines, and BG, DL, its tangents for these radii, respectively, CF : EK = AC : AE, or the sines of angle A for the radii AC, AE, or AB, AD, are proportional to these radii. The same may be proved for the tangents GB, LD, and the secants AG, AL. Hence, by equal ratios, CF : EK = GB : LD = AG : AL.

COR.—Hence, in a right-angled triangle AFC, since AC : CF = R : sin A, R · CF = AC · sin A, and if R = 1, CF = AC · sin A.

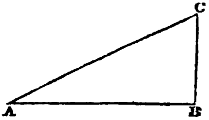
It is proved in the same way that AF = AC · cos A; and when AF is radius, FC = AF · tan A, and AC = AF · sec A.

Schol.—When two sides of a right-angled triangle are given, the third may be found by I. 47 and Cor.

PROPOSITION IV. THEOREM.

If any side of a right-angled triangle be made radius, the other two sides are proportional to the trigonometrical lines, which they represent for any other radius.

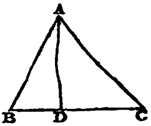
Let ABC be a right-angled triangle; when AC is radius, AB and BC represent the cosine and sine of A; and (Prop. III.) these are proportional to the cosine and sine for any other radius. Therefore if sin A, cos A, be the sine and cosine of A for any other radius, $CB : BA = \sin A : \cos A$; or $\sin A : \cos A = CB : BA$.



PROPOSITION V. THEOREM.

The sides of a plain triangle are to one another as the sines of the opposite angles.

From A, any angle in the triangle ABC, let AD be drawn perpendicular to BC; and because the triangle ABD is right-angled at D, $AB : AD = R : \sin B$; and, for the same reason, $AC : AD = R : \sin C$, and inversely, $AD : AC = \sin C : R$; therefore, by indirect equality, $AB : AC = \sin C : \sin B$. In the same manner, it may be demonstrated that $AB : BC = \sin C : \sin A$.

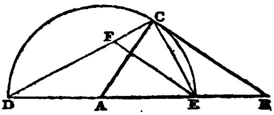


PROPOSITION VI. THEOREM.

The sum of two sides of a triangle is to their difference as the tangent of half the sum of the angles at the base to the tangent of half their difference.

Let ABC be any triangle, then if B and C denote the angles at B and C respectively, $AB + AC : AB - AC = \tan \frac{1}{2} (C + B) : \tan \frac{1}{2} (C - B)$.

From A as a centre, with the radius AC, describe the semicircle DCE; produce BA to D; join DC and CE, and draw EF parallel to BC. Then twice angle $AEC = AEC + ACE = DAC$ (I. 32) $= ABC + ACB = B + C$, therefore $AEC = \frac{1}{2} (C + B)$; and FEC =



ECB, and $\mathrm{AEC} - \mathrm{ECB} = \mathrm{B}$, the less angle for $\mathrm{AC} < \mathrm{AB}$; therefore (II. D. Cor. 2) ECB or $\mathrm{CEF} = \frac{1}{2}(\mathrm{C} - \mathrm{B})$. Also, $\mathrm{DB} = \mathrm{DA} + \mathrm{AB} = \mathrm{AB} + \mathrm{AC}$, and $\mathrm{BE} = \mathrm{AB} - \mathrm{AE} = \mathrm{AB} - \mathrm{AC}$. And since angle DCE is a right angle (III. 31), if EC be made radius, DC and FC are tangents of the angles AEC and FEC (Def. 4). But (VI. 2) $\mathrm{DB} : \mathrm{BE} = \mathrm{DC} : \mathrm{CF}$; or $\mathrm{AB} + \mathrm{AC} : \mathrm{AB} - \mathrm{AC} = \tan \frac{1}{2}(\mathrm{C} + \mathrm{B}) : \tan \frac{1}{2}(\mathrm{C} - \mathrm{B})$.

PROPOSITION VII. THEOREM.

If a perpendicular be drawn from the vertex upon the base of a triangle, the sum of the segments of the base is to the sum of the two sides as the difference between these sides to the difference between the segments of the base.

For (II. c. Cor. 1) the rectangle under the sum and difference of the sides is equal to that under the sum and difference of the segments of the base, therefore (VI. 16) the above proportion subsists.

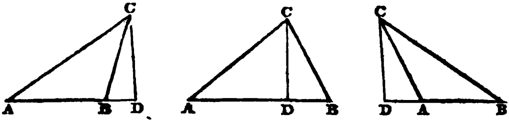
Schol.—The preceding propositions are sufficient for the solution of all the cases of trigonometry; but, in particular cases, some of the following propositions may be employed with advantage.

PROPOSITION VIII. THEOREM.

In any triangle, twice the rectangle contained by any two of its sides, is to the difference between the sum of their squares and the square of the other side, as radius to the cosine of the angle contained by these two sides.

Let ABC be any triangle, $2\,\mathrm{AB} \cdot \mathrm{BC}$ is to the difference between $\mathrm{AB}^2 + \mathrm{BC}^2$ and AC^2 as radius to cos B.

For, from C draw CD perpendicular to AB or AB produced. Then $\mathrm{BC} : \mathrm{BD} = \mathrm{R} : \cos \mathrm{B}$. But (VI. 1) $2\,\mathrm{AB} \cdot \mathrm{BC} : 2\,\mathrm{AB} \cdot \mathrm{BD} = \mathrm{BC} : \mathrm{BD}$; therefore $2\,\mathrm{AB} \cdot \mathrm{BC} : 2\,\mathrm{AB} \cdot \mathrm{BD} = \mathrm{R} : \cos \mathrm{B}$. But (II. 12, 13, or Ad. II. 15) $2\,\mathrm{AB} \cdot \mathrm{BD}$ is



the difference between $AB^2 + BC^2$ and AC^2; and hence the proposition is proved.

COR.—If the sides opposite to the angles A, B, C, be called a, b, c, respectively, and Radius $= 1$, then (Qu. X. and Ad. V. 7) $2ac : a^2 + c^2 - b^2 = 1 : \cos B$, or $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$. This applies only to the case in which the angle B is acute. When B is obtuse, then $\cos B = \frac{b^2 - (a^2 + c^2)}{2ac}$; but in this case, cos B ought to be negative, but $a^2 + c^2 - b^2$ is negative, for $b^2 > a^2 + c^2$; hence also in this case, $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ when its sign is considered.

Taking the side AC instead of AB, which is just changing b into c,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Also, if both sides be multiplied by $2ab$, $2ab \cos C = a^2 + b^2 - c^2$, and adding c^2 to both, and taking $2ab \cos C$ from them,

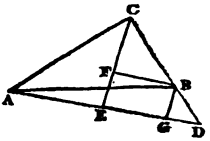
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

This applies also to both cases when C is acute and obtuse.

Schol.—When a, b, and c, are known, cos C may be found by the former expression; and when a, b, and C, are known, c may be found by the latter.

PROPOSITION IX. THEOREM.

In any triangle, the rectangle under the two sides, is to that under the excesses of the semiperimeter above these sides, as the square of the radius, to the square of the sine of half the vertical angle.



Let ABC be any triangle, produce CB till CD = CA; join AD, and draw CE bisecting the angle C, then (I. 4) CE is perpendicular to AD. Draw BF parallel to DE, and BG to CE. Then $AC : AE = \text{Rad} : \sin \frac{1}{2} C$

(Prop. IV.) if AC be radius; also $BC : BF$ or $EG = \text{Rad} : \sin \frac{1}{2} C$; hence (VI. 23, Cor.) $AC \cdot CB : AE \cdot EG = R^2 : \sin^2 \frac{1}{2} C$. But (II. c. Cor. 1) $AE \cdot EG = \frac{1}{2}(AB + BD) \frac{1}{2}(AB - BD)$ in the triangle ABD. But $AB + BD = AB + CD - CB = AB + AC - CB = AB + AC + CB - 2CB = 2(S - CB)$, if $S =$ semiperimeter. And $AB - BD = AB - (CD - CB) = AB - (AC - CB) = AB - AC + CB$ (Ad. II. 2) $= AB + AC + CB - 2AC = 2(S - AC)$; therefore $AC \cdot CB : (S - AC)(S - BC) = R^2 : \sin^2 \frac{1}{2} C$.

COR.—If the numerical values of the sides opposite to the angles A, B, C, be called a, b, c, respectively, and if $s =$ the semiperimeter and $R = 1$, then

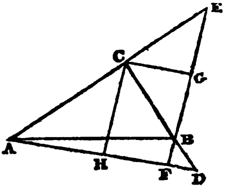
$$ab : (s - a)(s - b) = R^2 : \sin^2 \tfrac{1}{2} C.$$
$$\text{or } ab \cdot \sin^2 \tfrac{1}{2} C = (s - a)(s - b), \text{ or}$$
$$\sin^2 \tfrac{1}{2} C = \frac{(s - a)(s - b)}{ab}.$$

PROPOSITION X. THEOREM.

In any triangle, the rectangle under its two sides, is to that under the semiperimeter and its excess above the base, as the square of the radius, to the square of the cosine of half the vertical angle.

Let ABC be any triangle, and draw the figure ACDH as in the last; produce BF to meet AC produced in E, and draw CG parallel to AD.

Then angle $ACH = E$, and $DCH = CBE$, therefore $E = CBE$, and $CE = CB$. Also since CG is parallel to AD, the angles at G are right angles. And HG is a rectangle, therefore $CH = GF$. Now, $AC : CH$ or $GF = R : \cos \frac{1}{2} C$, and CE or $CB : EG = R : \cos E$ or $\cos \frac{1}{2} C$; therefore (VI. 23, Cor.) $AC \cdot CB : EG \cdot GF = R^2 : \cos^2 \frac{1}{2} C$. But (II. c. Cor. 1, and Ad. II. 15, Sch.) $4EG \cdot GF = (AE + AB)(AE - AB)$ in triangle ABE; and $AE + AB = AC + CB + AB = 2S$; and $AE - AB = AC + CB - AB = AC + CB + AB - 2AB = 2(S - AB)$, therefore $AC \cdot CB : S(S - AB) = R^2 : \cos^2 \frac{1}{2} C$.



COR.—Take a, b, c, &c., as in Cor. to last proposition, and $ab : s\,(s-c) = R^2 : \cos^2 \frac{1}{2} C$; or if $R = 1$, $ab \cdot \cos^2 \frac{1}{2} C = s\,(s-c)$, or

$$\cos^2 \tfrac{1}{2} C = \frac{s\,(s-c)}{ab}.$$

PROPOSITION XI. THEOREM.

In any triangle, the rectangle under the semiperimeter and its excess above the base, is to the rectangle under its excesses above the two sides, as the square of the radius, to the square of the tangent of half the vertical angle.

For (Prop. X.) $AC \cdot CB : S\,(S - AB) = R^2 : \cos^2 \frac{1}{2} C$, and (Prop. IX.) $AC \cdot CB : (S - AC)\,(S - BC) = R^2 : \sin^2 \frac{1}{2} C$, therefore, by direct equality, $S\,(S - AB) : (S - AC)\,(S - BC) = \cos^2 \frac{1}{2} C : \sin^2 \frac{1}{2} C$. But (Cor. 7 to Definitions, and VI. 22, Cor.) $\cos^2 \frac{1}{2} C : \sin^2 \frac{1}{2} C = R^2 : \tan^2 \frac{1}{2} C$, therefore $S\,(S - AB) : (S - AC)\,(S - BC) = R^2 : \tan^2 \frac{1}{2} C$.

COR.—Therefore $s\,(s-c) : (s-a)\,(s-b) = R^2 : \tan^2 \frac{1}{2} C$; and when $R = 1$, $s\,(s-c)\, \tan^2 \frac{1}{2} C = (s-a)\,(s-b)$, or

$$\tan^2 \tfrac{1}{2} C = \frac{(s-a)\,(s-b)}{s\,(s-c)}.$$

EXERCISES.

1. If in any triangle a perpendicular be drawn from the vertex upon the base, the segments of the base have the same ratio, as the tangents of the parts into which the vertical angle is divided.

2. The base of a triangle is to the sum of its two sides, as the cosine of half the sum of the angles at the base to the cosine of half their difference.

3. The base of a triangle is to the difference of its sides, as the sine of half the sum of the angles at the base to the sine of half their difference.

4. The base of a triangle is to the difference of its segments, as the sine of the vertical angle to the sine of the difference of the angles at the base.

5. Half the perimeter of a triangle is to its excess above

the base, as the cotangent of half either of the angles at the base to the tangent of half the other angle.

6. The excess of half the perimeter of a triangle above the less side is to its excess above the greater, as the tangent of half the greater angle at the base to the tangent of half the less.

7. In a right-angled triangle, radius is to the sine of double one of the acute angles, as the square of half the hypotenuse to the area of the triangle.

8. Radius is to the tangent of half the vertical angle of a triangle, as the rectangle under half the perimeter and its excess above the base to the area of the triangle.

ALGEBRAICAL PRINCIPLES.

I. In Algebra the values of quantities are represented by letters. These values are numbers which denote the number of times that the unit of measure of each kind of quantities is contained in them respectively. These numbers, therefore, may be either terminate or interminate. The algebraical principles employed in the preceding treatise are very simple, and have been already explained, except a few that are referred to in the additional fifth book.

The student may illustrate the propositions in the second book, by taking numbers and products instead of lines and rectangles, and the squares of numbers instead of the squares of lines. A small ^{2} placed over a number or letter means its square or the product arising from multiplying it by itself.

1. If $m = 3$, $m^2 = mm$, and $3^2 = 3 \times 3 = 9$.

2. If A be a line $= 3$ inches, $A^2 = 3^2$ square inches $= 3 \times 3$ square inches $= 9$ square inches, or 9 times the square described on a line of 1 inch. In this case, 1 inch is the unit of measure.

3. Let $A = 5$ feet, and m a number $= 3$, then $mA = 3 \times 5$ feet $= 15$ feet.

4. In the fourth proposition of the second book, let $AB = 8$, $AC = 5$, $CB = 3$, then $AB^2 = AC^2 + CB^2 +$

$2\text{AC}\cdot\text{CB}$, or $8^2 = 5^2 + 3^2 + 2 \times 5 \times 3 = 25 + 9 + 30 = 64$, and $8^2 = 64$.

The propositions of the fifth book may also be illustrated by numbers.

If $\text{A} = 12$, $\text{B} = 4$, $\text{C} = 18$, $\text{D} = 6$, then $\text{A} : \text{B} = \text{C} : \text{D}$ or $12 : 4 = 18 : 6$. Also $\text{A} + \text{B} : \text{B} = \text{C} + \text{D} : \text{D}$ or $12 + 4 : 4 = 18 + 6 : 6$ or $16 : 4 = 24 : 6$.

Also $\frac{\text{A}}{\text{B}} = \frac{\text{C}}{\text{D}}$, or $\frac{12}{4} = \frac{18}{6} = 3$.

II. The following are examples of products and quotients, and simple cases of fractions :—

If $m = 3$, and $n = 5$, then,

1. $mn = 3 \times 5 = 15$.
2. $mn^2 = 3 \times 5^2 = 3 \times 25 = 75$.
3. $\frac{mn^2}{n} = \frac{mnn}{n} = mn$, for mn multiplied by n, and divided by n, is just mn; or $\frac{3\times 5^2}{5} = \frac{3\times 5\times 5}{5} = 3\times 5 = 15 = mn$.
4. $\frac{mn}{n^2} = \frac{mn}{nn} = \frac{m}{n}$, or $\frac{3\times 5}{5^2} = \frac{3\times 5}{5\times 5} = \frac{3}{5} = \frac{m}{n}$.

If also $p = 2$, and $q = 4$, then,

5. $mnpq = 3\times 5 \times 2 \times 4 = 120$; or $mpnq = 3\times 2\times 5 \times 4 = 120$.
6. $\frac{mnpq}{q} = mnp$, or $\frac{3\times 5\times 2\times 4}{4} = 3\times 5\times 2 = 30 = mnp$.
7. $\frac{mnpq}{nq} = \frac{mp\cdot nq}{nq} = mp$, or $\frac{3\times 2\times 5\times 4}{5\times 4} = 3 \times 2 = 6 = mp$.
8. $\frac{mnp}{n^2} = \frac{mnp}{nn} = \frac{mp}{n}$, or $\frac{3\times 5\times 2}{5^2} = \frac{3\times 5\times 2}{5\times 5} = \frac{3\times 2}{5} = \frac{6}{5} = \frac{mp}{n}$.
9. $\frac{mnpq}{n^2q^2} = \frac{mp\cdot nq}{nq\cdot nq} = \frac{mp}{nq}$, or $\frac{3\times 5\times 2\times 4}{5^2\times 4^2} = \frac{3\times 2}{5\times 4} = \frac{6}{20} = \frac{mp}{nq}$.

The rules for numerical vulgar fractions apply to the simple algebraical fractions treated of here.

The three following articles are propositions respecting common fractions:—

III. Let $\frac{a}{b} = m$; a, b, and m, being integers, then $\frac{b}{a} = \frac{1}{m}$.

If $a = 8$, and $b = 2$, then $m = \frac{a}{b} = \frac{8}{2} = 4$; and $\frac{b}{a} = \frac{2}{8} = \frac{1}{4} = \frac{1}{m}$.

IV. If a, b, and m, be integers, then $\frac{a}{b} = \frac{ma}{mb}$.

Let $a = 6$, $b = 2$, and $m = 3$, then $\frac{a}{b} = \frac{6}{2} = 3$; and $\frac{ma}{mb} = \frac{3 \times 6}{3 \times 2} = \frac{18}{6} = 3$; or $\frac{a}{b} = \frac{ma}{mb}$.

V. Let a, b, c, and d, be integers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

If $a = 6$, $b = 3$, $c = 8$, and $d = 5$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{6}{3} \times \frac{8}{5} = 2 \times 1\frac{3}{5} = 3\frac{1}{5}$; and $\frac{ac}{bd} = \frac{6 \times 8}{3 \times 5} = \frac{48}{15} = 3\frac{1}{5}$; or $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

VI. The propositions in the last three articles are also true when a, b, c, d, and m, are fractional terminate numbers.

For example, let $a = \frac{e}{f}$, $b = \frac{g}{h}$ and $m = \frac{k}{l}$, the numbers e, f, g, h, k, and l, being integers; then $\frac{a}{b} = \frac{ma}{mb}$.

For $ma = \frac{k}{l} \cdot \frac{e}{f} = \frac{ke}{lf}$, and $mb = \frac{k}{l} \cdot \frac{g}{h} = \frac{kg}{lh}$, therefore

$\frac{ma}{mb}=\frac{ke}{lf}\cdot\frac{lh}{kg}=\frac{eh}{fg}$. And $\frac{a}{b}=\frac{e}{f}\cdot\frac{h}{g}=\frac{eh}{fg}$; therefore $\frac{a}{b}=\frac{ma}{mb}$.

As a numerical illustration, let $a=\frac{2}{3}$, $b=\frac{7}{5}$, $m=\frac{9}{11}$. Then $ma=\frac{9}{11}\times\frac{2}{3}=\frac{6}{11}$; $mb=\frac{9}{11}\times\frac{7}{5}=\frac{63}{55}$; therefore $\frac{ma}{mb}=\frac{6}{11}\times\frac{55}{63}=\frac{10}{21}$. Also $\frac{a}{b}=\frac{2}{3}\times\frac{5}{7}=\frac{10}{21}$. Therefore $\frac{a}{b}=\frac{ma}{mb}$.

In a similar manner, the propositions in Art. III. and V. may be proved.

The same propositions are true when a, b, c, d, and m, are interminate numbers. Their demonstrations are given in the four following articles. The accented letter in every case denotes an interminate number greater than *that* denoted by the same letter unaccented, and the same letter doubly accented denotes a terminate number intermediate between the other two. Thus $a' > a$, $a'' < a'$, and $a'' > a$; where a'' is the intermediate terminate number. When a contradictory conclusion is arrived at on the hypothesis that $a' > a$, it is of course false, and it may in each case be similarly proved that the hypothesis of $a' < a$ is also false; and hence if the first hypothesis be proved false in any case, it may be concluded that a' must $= a$. These remarks will prevent unnecessary repetitions.

VII. Let a, b, and m, be one or all of them interminate, and $\frac{a}{b}=m$, then $\frac{b}{a}=\frac{1}{m}$. The values of $\frac{b}{a}$ and $\frac{1}{m}$ are each supposed to be expressed by a decimal fraction, and the proposition asserts the equality of these decimals.

1. When a, and therefore m, is interminate.

Let $\frac{b}{a}=\frac{1}{m'}$. Then $\frac{b}{a''}=\frac{1}{m''}$ (VI.); but $\frac{1}{m''}>\frac{1}{m'}$, for

$m'' < m'$; therefore $\frac{b}{a''} > \frac{b}{a}$, or $a'' < a$. But $\frac{a''}{b} = m''$ (VI.), and $m'' > m$, therefore $\frac{a''}{b} > \frac{a}{b}$, or $a'' > a$, and it was also shown to be less, therefore $\frac{b}{a} = \frac{1}{m}$.

2. When b, and therefore m, is interminate.
The proof is similar to the last.
3. When a, b, and m, are interminate.

Let $\frac{b'}{a} = \frac{1}{m}$; and also let $\frac{b''}{a} = \frac{1}{n}$. Then (1st case) $\frac{a}{b''} = n$; but $\frac{a}{b} = m$, therefore $n < m$, for $b'' > b$. But $b' > b''$, and therefore $\frac{1}{m} > \frac{1}{n}$, or $n > m$, and it was also shown to be less; hence $\frac{b}{a} = \frac{1}{m}$.

Cor.—If $\frac{a}{b} = \frac{c}{d} = m$, then $\frac{b}{a} = \frac{d}{c}$.

VIII. When a, b, and m, are, one or all, interminate, $\frac{a}{b} = \frac{ma}{mb}$.

1. When a is interminate.

Let $\frac{a}{b} = \frac{ma'}{mb}$. Then $\frac{a''}{b} = \frac{ma''}{mb}$ (VI.); but $\frac{ma''}{mb} < \frac{ma'}{mb}$, therefore $\frac{a''}{b} < \frac{a}{b}$, and hence $a'' < a$; but $a'' > a$; therefore $\frac{a}{b} = \frac{ma}{mb}$.

2. When b is interminate.
This is proved like the last case.
3. When a and b are interminate.

Let $\frac{a}{b} = \frac{ma'}{mb}$. Then $\frac{a''}{b} = \frac{ma''}{mb}$ (2d case); but $\frac{ma''}{mb} <$

$\frac{ma'}{mb}$, therefore $\frac{a''}{b} < \frac{a}{b}$, or $a'' < a$; but $a'' > a$; hence $\frac{a}{b} = \frac{ma}{mb}$.

4. When m is interminate.

Let $\frac{a}{b} = \frac{m'a}{mb}$. Then $\frac{a}{b} = \frac{m''a}{m''b}$ (VI.); and $m''a < m'a$, and $m''b > mb$, therefore $\frac{m''a}{m''b} < \frac{m'a}{mb}$, but these are equal, for they are each $= \frac{a}{b}$, therefore $\frac{a}{b} = \frac{ma}{mb}$.

5. When a and m are interminate.

Let $\frac{a}{b} = \frac{ma'}{mb}$. Then $\frac{a''}{b} = \frac{ma''}{mb}$ (4th case); but $\frac{ma''}{mb} < \frac{ma'}{mb}$, therefore $\frac{a''}{b} < \frac{a}{b}$, and $a'' < a$; but $a'' > a$; therefore $\frac{a}{b} = \frac{ma}{mb}$.

6. When b and m are interminate.

The proof is similar to the last.

7. When a, b, and m, are interminate.

Let $\frac{a}{b} = \frac{ma'}{mb}$. Then $\frac{a''}{b} = \frac{ma''}{mb}$ (6th case); but $\frac{ma''}{mb} < \frac{ma'}{mb}$, or $\frac{a''}{b} < \frac{a}{b}$, and hence $a'' < a$; but $a'' > a$; therefore $\frac{a}{b} = \frac{ma}{mb}$.

IX. Let a, b, c, and d, be one or all interminate, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

1. When one of them as a is interminate.

Let $\frac{a}{b} \cdot \frac{c}{d} = \frac{a'c}{bd}$. Then $\frac{a''}{b} \cdot \frac{c}{d} = \frac{a''c}{bd}$ (VI.); but $\frac{a''c}{bd} <$

$\frac{a'c}{bd}$, therefore $\frac{a''}{b}\cdot\frac{c}{d} \angle \frac{a}{b}\cdot\frac{c}{d}$, and hence $a'' \angle a$; but $a'' \gt a$, therefore $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.

The proof is similar when any other of the numbers is interminate.

2. When any two of the numbers, as a and d, are interminate.

Let $\frac{a}{b}\cdot\frac{c}{d}=\frac{a'}{b}\cdot\frac{c}{d}$. Then $\frac{a''}{b}\cdot\frac{c}{d}=\frac{a''c}{bd}$ (1st case), and $\frac{a''c}{bd} \angle \frac{a'c}{bd}$, therefore $\frac{a''}{b}\cdot\frac{c}{d} \angle \frac{a}{b}\cdot\frac{c}{d}$, or $a'' \angle a$; but $a'' \gt a$; hence $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.

The proof is similar when any other two are interminate. The third case, when three are interminate, may be similarly proved by referring to the second case; and the fourth case, when they are all interminate, is similarly proved by referring to the third case.

X. The second property of numbers stated in the scholium after the definitions of the additional fifth book, may be proved thus. Let $\frac{m}{n}$ be any vulgar fraction. Reduce $\frac{1}{n}$ to a decimal, and it is obvious, that, since the remainders cannot exceed $n-1$, after the first $n-1$ terms of the quotient are found, the remainder, if the decimal is not finite, must be the same as some of the preceding remainders, or else 1; and hence the decimal will be a circulate, the numbers of figures in which will not exceed $n-1$. Then the product of this circulate by m, will be a circulate consisting of the same number of figures, and its value $=\frac{m}{n}$.

The tenth proposition of the book on the quadrature of the circle, may be easily proved by means of two of the propositions in the additional fifth book:

Since $A \cdot B : M^2 = (A : M, B : M)$ (VI. 23), and $(A : M, B : M) = (a : 1, b : 1)$ (Ad. V. 7) $= \frac{a}{1} \cdot \frac{b}{1}$ (Ad. V. 26) $= ab$; therefore $A \cdot B : M^2 = ab$, or $A \cdot B = abM^2$. In this demonstration a and b may be either terminate or interminate.

The cause for introducing the new terms, terminate and interminate, is that the terms, rational and irrational, are not sufficiently general; for the term irrational, applied to numbers, properly comprehends only that class of interminate numbers, that results from extracting the roots of numbers that are not complete powers.

THE END.

Edinburgh: Printed by W. and R. Chambers, 19, Waterloo Place.